Hasimoto frames and the Gibbs measure of the periodic nonlinear Schrödinger equation

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The paper interprets the cubic nonlinear Schrödinger equation as a Hamiltonian system with infinite dimensional phase space. There exists a Gibbs measure which is invariant under the flow associated with the canonical equations of motion. The logarithmic Sobolev and concentration of measure inequalities hold for the Gibbs measures, and here are extended to the k-point correlation function and distributions of related empirical measures. By Hasimoto's theorem, NLSE gives a Lax pair of coupled ODE for which the solutions give a system of moving frames. The paper studies the evolution of the measure induced on the moving frames by the Gibbs measure; the results are illustrated by numerical simulations. The paper contains quantitative estimates with well-controlled constants on the rate of convergence of the empirical distribution in Wasserstein metric.

I. INTRODUCTION

Consider the Hamiltonian

$$H_3 = \frac{1}{2}\int_{\mathbb{T}}\Big(\Big(\frac{\partial P}{\partial x}\Big)^2 + \Big(\frac{\partial Q}{\partial x}\Big)^2\Big)dx + \frac{\beta}{2\gamma}\int_{\mathbb{T}}(P^2+Q^2)^{\gamma}dx, \tag{1.1}$$

on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ which gives the canonical equations of motion

$$\begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}\frac{\partial}{\partial t}\begin{bmatrix} Q\\ P\end{bmatrix} = -\frac{\partial^2}{\partial x^2}\begin{bmatrix} Q\\ P\end{bmatrix} + \beta(P^2+Q^2)^{\gamma-1}\begin{bmatrix} Q\\ P\end{bmatrix} \tag{1.2}$$

so $u = P+iQ$ satisfies the nonlinear Schrödinger equation

$$i\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} + \beta|u|^{2(\gamma-1)}u. \tag{1.3}$$

When $\gamma = 2$, we have the cubic nonlinear Schrödinger equation. The spatial variable is $x\in\mathbb{T}$, and the functions are periodic, so that the system applies to fields parametrized by a circle. Throughout the paper, we write (M,d,μ) for a complete and separable metric space with a Radon (inner regular) probability measure μ on the σ-algebra generated by the Borel subsets. The squared L^2 norm $H_1 = \int(P^2+Q^2)$ is formally invariant under the canonical equations of motion, so we can consider possible invariant measures on

$$B_K = \Big\{u = P+iQ : P,Q\in L^2(\mathbb{T};\mathbb{R}) : \int_{\mathbb{T}}(P^2+Q^2)dx \le K\Big\}. \tag{1.4}$$

The Gibbs measure on B_K for this micro-canonical ensemble is

$$\mu_{K,\beta}(du) = Z_K(\beta)^{-1}\mathbb{I}_{B_K}(u)\exp\Big(-\frac{\beta}{4}\int_{\mathbb{T}}|u|^4dx\Big)W(du) \tag{1.5}$$

where $W(du)$ is Wiener loop measure, $\mathbb{I}_{B_K}$ is the indicator function of B_K and $Z_K(\beta)$ is a normalizing constant. Lebowitz, Rose and Speer[31] proved existence of such an invariant measure, so that for all $K>0$ and $\beta\in\mathbb{R}$ there exists $Z_K(\beta)>0$ such that $\mu_{K,\beta}$ is a Radon probability measure on $B_K\subset L^2(\mathbb{T};\mathbb{R}^2)$. When $\beta=0$, we refer to the measure as free Wiener loop measure, indicating that the dynamics are free of potentials. For $\beta<0$, (1.3) is said to be focussing and the Hamiltonian is unbounded below, giving the source of the technical problem, which is addressed by restricting the measure to B_K.

Bourgain[12] gave an alternative existence proof using random Fourier series, and showed that the measure is invariant under the flow in the sense that the Cauchy problem is well posed on the support. Further refinements include a result of McKean[36], that the sample paths are Hölder continuous, and a result from Theorem 1.2(iv) in Ref.[9] that the invariant measure of the microcanonical ensemble satisfies a logarithmic Sobolev inequality. Random Fourier series fit naturally into Sturm's theory of metric measure spaces, which we use to reduce some of the analysis to invariant measures on finite-dimensional Hamiltonian systems.

The focusing case for spatial variable $x \in \mathbb{R}$ captures soliton solutions, and Ref.[31] discuss the possible transition of the system between an ambient bounded random field and a soliton solution. For $x \in \mathbb{T}$, the notion of a spatially localized solution is inapplicable, but some of the results are still relevant[31].

Bourgain[13] page 128 comments that invariant Gibbs measures for the periodic cubic Schrödinger equation can be constructed on other phase spaces, and one can consider Gibbs measures on L^2 that have different normalizations than $\mu_{K,\beta}$.

In section two, we consider tensor products of Hilbert space H and a k-point density matrix. For μ_0 a centered Gaussian measure on H, we express a specific integral

$$J^{(k)} = \int_H |u^{\otimes k}\rangle\langle u^{\otimes k}|\, \mu_0(du)$$

as a series of elementary tensors. This calculation involves combinatorial results which are expressed in terms of Knuth's odd and even decompositions of Young diagrams. In section three, we use concentration of measure results to show how $|u^{\otimes k}\rangle\langle u^{\otimes k}|$ is close to its mean value $J^{(k)}$ on a set of large probability. This statement also holds when we replace μ_0 by the Gibbs measure.

In section 4 we introduce metric probability measure spaces and show how the infinite-dimensional dynamical system (1.2) can be approximated by finite-dimensional dynamical systems, particularly involving random Fourier series. In particular, we show that $x \mapsto u(x,t)$ is γ-Hölder continuous for $0 < \gamma < 1/16$, in the sense that $\sup\{\|u(x+h,t) - u(x,t)\|_{L^4_x}/|h|^\gamma); h \neq 0\}$ is almost surely finite, for fixed $0 < t_0 < t$.

We also obtain results on the empirical distributions that arise when we sample solutions of (1.2) with respect to Gibbs measure (1.5), which we use in the numerical experiments in section 7.

Hasimoto[25] observed that (1.2) can be expressed as a Lax pair of coupled ordinary differential equations with solutions in $SO(3)$, one of which is the Serret-Frenet system for a moving frame on a curve in $\mathbb{R}^3$. Cruzeiro and Malliavin[17] developed stochastic differential geometry for frames, pursuing Cartan's precedent[15]. In sections 5 and 6 we consider the evolution of the dynamical system corresponding to Hasimoto frames under the Gibbs measure. In section 7 we present numerical experiments regarding the solutions, which illustrate the nature of frames that arise from the solutions of (1.2) for typical elements in the support of the Gibbs measure (1.5). In the appendix[25], Hasimoto expressed the change of variables in a polar decomposition $u = \sqrt{\rho}\exp(i\phi)$ where ρ is a probability density and ϕ a phase, and derived Betchov's intrinsic equation for vortex filaments from the nonlinear Schrödinger equation. We remark that Villani[42] page 691 carries out a similar a transformation to interpret the linear Schrödinger equation as a transport problem for densities ρ for a suitable action integral. The current paper is a further step at introducing transportation methods into PDE.

II. TENSOR PRODUCTS AND k-POINT DENSITY MATRICES FOR GAUSSIAN MEASURE

Let H be a separable complex Hilbert space, with inner product $\langle \cdot \mid \cdot \rangle$ which is linear in the second argument. The algebraic tensor product $H \otimes H$ is identified with the set of finite-rank operators on H, and then we identify the injective tensor product $H \check{\otimes} H$ with the algebra $\mathscr{L}(H)$ of bounded linear operators on H and the projective tensor product $H \hat{\otimes} H$ with the ideal $\mathscr{L}^1(H)$ of trace class operators on H. By the theory of metric tensor products the dual space of $\mathscr{L}^1(H)$ is canonically $\mathscr{L}(H)$. For $H = L^2$, the identification is

$$f \otimes \bar{g} = |\, f\rangle\langle g\,| : h \mapsto f(x) \int \bar{g}(y)h(y)dy.$$

Let $A \in \mathscr{L}^1(H)$ be self-adjoint such that $0 \leq A \leq I$, and let μ_0 be a Gaussian measure on H of mean zero and covariance A. By the spectral theorem, We can choose an orthonormal basis $(\varphi_j)_{j=1}^\infty$ of H such that $A\varphi_j = \alpha_j \varphi_j$ where the spectrum of A is the closure of $\{\alpha_j : j = 1, 2, \dots\}$. Then we introduce mutually independent Gaussian $N(0,1)$ random variables $(\gamma_j)_{j=1}^\infty$ and the vector

$$u = \sum_{j=1}^\infty \sqrt{\alpha_j}\gamma_j \varphi_j \tag{2.1}$$

so that μ_0 is the distribution of u on H, as one easily checks by computing the expectation

$$\begin{aligned}\mathbb{E}\exp(i\langle f,u\rangle) &= \mathbb{E}\exp\Big(\sum_j i\sqrt{\alpha_j}\langle f,\varphi_j\rangle\gamma_j\Big)\\ &= \exp\Big(\sum_j -\frac{1}{2}\alpha_j\langle f,\varphi_j\rangle^2\Big)\\ &= \exp\big(-\frac{1}{2}\langle Af,f\rangle\big) \qquad (f\in H).\end{aligned} \tag{2.2}$$

Hence A is the mean of rank-one tensors with respect to Gaussian measure

$$A=\int_H \mid u\rangle\langle u\mid \mu_0(du).$$

The k-fold tensor product $H^{\otimes k}$ can be completed to give a Hilbert space, so that the space $H^{s\otimes k}$ of symmetric tensors gives a closed linear subspace. We consider

$$J^{(k)}=\int_H |u^{\otimes k}\rangle\langle u^{\otimes k}|\,\mu_0(du).$$

An element of $H^{\otimes k}\hat{\otimes}H^{\otimes k}$ determines a linear operator $\mathscr{L}(H^{\otimes k})$, commonly referred to as a matrix, so $J^{(k)}\in (L^2)^{s\otimes k}\hat{\otimes}(L^2)^{\otimes k}$ gives a k-point density matrix, or equivalently a trace class operator $J^{(k)}\in\mathscr{L}^1(H^{s\otimes k})$. The following computation of Gaussian moments is known in Quantum field theory as Wick's theorem.

Lemma 3.3 of[32] contains calculations regarding $J^{(k)}$ which we have not been able to interpret, particularly line 8 of page 79. Here we calculate $J^{(2)}$ directly, before addressing the case of general k. Evidently we have $\mathbb{E}(\gamma_j\gamma_\ell\gamma_m\gamma_n)=0$ if one of the indices j,ℓ,m,n is distinct from all the others; otherwise, we have all the indices equal, or two distinct pairs of equal indices. Hence we have

$$\begin{aligned}\int_H \mid u\otimes u\rangle\langle u\otimes u\mid \mu_0(du) &= \sum_{j,\ell,m,n}\sqrt{\alpha_j\alpha_\ell\alpha_m\alpha_n}\mid\varphi_j\otimes\varphi_\ell\rangle\langle\varphi_m\otimes\varphi_n\mid\mathbb{E}(\gamma_j\gamma_\ell\gamma_m\gamma_n)\\ &= \sum_j 3\alpha_j^2\mid\varphi_j\otimes\varphi_j\rangle\langle\varphi_j\otimes\varphi_j\mid\\ &\quad+\sum_{j,\ell:j\neq\ell}\alpha_j\alpha_\ell\mid\varphi_j\otimes\varphi_\ell\rangle\langle\varphi_j\otimes\varphi_\ell\mid\\ &\quad+\sum_{j,m:j\neq m}\alpha_j\alpha_m\mid\varphi_j\otimes\varphi_j\rangle\langle\varphi_m\otimes\varphi_m\mid\\ &\quad+\sum_{j,\ell:j\neq\ell}\alpha_j\alpha_\ell\mid\varphi_j\otimes\varphi_\ell\rangle\langle\varphi_\ell\otimes\varphi_j\mid\end{aligned} \tag{2.3}$$

and we combine the second and fourth of these to obtain

$$\begin{aligned}\int_H \mid u\otimes u\rangle\langle u\otimes u\mid \mu_0(du) &= \sum_j 3\alpha_j^2\mid\varphi_j\otimes\varphi_j\rangle\langle\varphi_j\otimes\varphi_j\mid\\ &\quad+\frac{1}{2}\sum_{j,\ell:j\neq\ell}\alpha_j\alpha_\ell\mid\varphi_j\otimes\varphi_\ell+\varphi_\ell\otimes\varphi_j\rangle\langle\varphi_j\otimes\varphi_\ell+\varphi_\ell\otimes\varphi_j\mid\\ &\quad+\sum_{j,m:j\neq m}\alpha_j\alpha_m\mid\varphi_j\otimes\varphi_j\rangle\langle\varphi_m\otimes\varphi_m\mid\end{aligned} \tag{2.4}$$

which exhibits the right-hand side as a symmetric tensor, in which the final term shows the integral is not diagonal with respect to the orthonormal basis

$$\Big\{\varphi_j\otimes\varphi_j,\quad (\varphi_j\otimes\varphi_\ell+\varphi_\ell\otimes\varphi_j)/\sqrt{2};\quad j,\ell\in\mathbb{N}; j\neq\ell\Big\}$$

of the symmetric tensor product $H\otimes_s H$, hence $J^{(2)}$ is not a multiple of $A\otimes A$.

For $k\in\mathbb{N}$, let Π_k be the set of all partitions of k so that $\pi\in\Pi_k$ may be expressed as $k=k_1+k_2+\dots+k_n$ where the row lengths $k_j\in\mathbb{N}$ have $k_1\geq k_2\geq\dots\geq k_n$. Given such π and a n-element subset $\{j_1,\dots,j_n\}$ of $\mathbb{N}$, there is a symmetric tensor

$$\frac{1}{\sqrt{n!}}\sum_\sigma \varphi_{\sigma(j_1)}^{\otimes k_1}\otimes\dots\otimes\varphi_{\sigma(j_n)}^{\otimes k_n}\in H^{\otimes k}$$

where the sum is over all the permutations σ of $\{j_1,\dots,j_n\}$. The set of all such tensors gives an orthonormal basis of the k-fold symmetric tensor product $H^{s\otimes k}$.

We express u as in (2.1) and consider the expansion

$$|\, u^{\otimes k}\rangle\langle u^{\otimes k}\,| \;=\; \sum_{(m_1,\dots,m_{2k})\in\mathbb{N}^{2k}} \sqrt{\alpha_{m_1}\dots\alpha_{m_{2k}}}\gamma_{m_1}\dots\gamma_{m_{2k}}\,|\,\varphi_{m_1}\otimes\dots\otimes\varphi_{m_k}\rangle\langle\varphi_{m_{k+1}}\otimes\dots\otimes\varphi_{m_{2k}}\,| \tag{2.5}$$

in terms of this orthonormal basis of $H^{s\otimes k}$, and look for the terms that do not vanish after integration with respect to μ_0.

Definition II.1. (even decomposition) Given $\pi\in\Pi_k$ consider a pair $(\lambda,\rho)\in\Pi_k^2$ with rows $\lambda: k=\ell_1+\ell_2+\dots+\ell_n$ where $\ell_j\in\mathbb{N}\cup\{0\}$ and $\rho: k=r_1+r_2+\dots+r_n$ where $r_j\in\mathbb{N}\cup\{0\}$ and

$$2k_j=\ell_j+r_j \qquad (j=1,\dots,n),$$

so that λ and ρ have equal numbers of odd rows; here rows may have zero lengths, and the rows are not necessarily in decreasing order. We refer to (λ,ρ) as an even decomposition of π.

Remark II.2. There are various alternative descriptions of even decompositions. We write $\lambda\sim\rho$ if λ and ρ are partitions that have equal numbers of boxes and equal numbers of odd rows; evidently $\sim$ is an equivalence relation on the set of partitions. By[29] Theorem 4 there is a bijection between symmetric matrices A that have entries in $\mathbb{N}\cup\{0\}$ with column sums $c_1,\dots,c_n$ and Young tableaux P such that have c_j occurrences of j as entries and number of columns of P of odd length equals the trace of A. Given symmetric matrices A and B with entries in $\mathbb{N}\cup\{0\}$ such that A and B have equal traces and equal totals of entries, then the *RSK* correspondence takes A to P and B to Q where P and Q are Young tableaux with an equal number of boxes, and their transposed diagrams P' and Q' have an equal number of odd rows, so $P'\sim Q'$.

For notational convenience, we also regard $\varphi_j^{\otimes 0}$ as a factor which may be omitted in tensor products. Then given such a triple (π,λ,ρ) and an n-subset $\{j_1,\dots,j_n\}$ of $\mathbb{N}$,

$$\alpha_{j_1}^{k_1}\dots\alpha_{j_n}^{k_n}\,|\,\varphi_{j_1}^{\otimes\ell_1}\otimes\dots\otimes\varphi_{j_n}^{\otimes\ell_n}\rangle\langle\varphi_{j_1}^{\otimes r_1}\otimes\dots\otimes\varphi_{j_n}^{\otimes r_n}\,|\,\mathbb{E}\big(\gamma_{j_1}^{2k_1}\gamma_{j_2}^{2k_2}\dots\gamma_{j_n}^{2k_n}\big) \tag{2.6}$$

where

$$\mathbb{E}\big(\gamma_{j_1}^{2k_1}\gamma_{j_2}^{2k_2}\dots\gamma_{j_n}^{2k_n}\big)=\prod_{j=1}^{n}\frac{(2k_j)!}{2^{k_j}k_j!} \tag{2.7}$$

gives a nonzero summand in $J^{(k)}$.

Conversely, let $(\lambda,\rho)\in\Pi_k^2$ and suppose that λ and ρ have equal numbers of odd rows, so that after adding zero rows and reordering the rows we have $r_j+\ell_j$ even for all j. Then we introduce $2k_j=\ell_j+r_j$ and after a further reordering write $k=k_1+k_2+\dots+k_n$ where $k_j\in\mathbb{N}$ have $k_1\geq k_2\geq\dots\geq k_n$, and we have $\pi\in\Pi_k$ as above. Given a n-subset $\{j_1,\dots,j_n\}$ of $\mathbb{N}$, we take $2k_m$ copies of j_m and split them as ℓ_m on the bra side and r_m on the ket side of the tensor for $m=1,\dots,n$, making a contribution as in (2.6). We summarize these results as follows.

Proposition II.3. *The integral $J^{(k)}$ is the sum over the summands (2.6) that arise from a $\pi\in\Pi_k$ with n nonzero rows, a n-subset of $\mathbb{N}$, and an even decomposition of π into a pair $(\lambda,\rho)\in\Pi_k^2$ where λ and ρ have equal numbers of odd rows.*

III. CONCENTRATION OF k-POINT MATRICES FOR GIBBS MEASURE

Let $H=L^2(\mathbb{T};\mathbb{R})$ and let $(\gamma_j)_{j\in\mathbb{Z}}$ be mutually independent Gaussian $N(0,1)$ random variables on some probability space (Ω,P), where $z_j=(\gamma_j+i\gamma_{-j})/\sqrt{2})$ and $z_{-j}=(\gamma_j-i\gamma_{-j})/\sqrt{2}$ for $j\in\mathbb{N}$. Then we take Brownian loop to be the random Fourier series in the style

$$u(\theta)=\sum_{j\in\mathbb{Z}\setminus\{0\}}\frac{z_je^{ij\theta}}{|j|}, \tag{3.1}$$

so that Wiener loop $W(du)$ is the distribution of $u\in H$, namely the probability measure induced by random variable u via $(\Omega,P)\to(H,W)$. By orthogonality, we have

$$\int_{\mathbb{T}}|u(\theta)|^2\frac{d\theta}{2\pi}=\sum_{j\in\mathbb{Z}\setminus\{0\}}\frac{|z_j|^2}{j^2} \tag{3.2}$$

so that $u \in B_{K/2\pi}$ if and only if $\sum_j \gamma_j^2/j^2 \le K$. Chernoff's inequality, and independence we have

$$\mathbb{P}\left[\sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{\gamma_j^2}{j^2} \ge K\right] \le e^{-tK}\frac{\pi\sqrt{2t}}{\sin(\pi\sqrt{2t})} \qquad (0<t<1/2, K>0). \tag{3.3}$$

The low Fourier modes are the predominant terms since one has the estimate

$$\mathbb{P}\left[\sum_{j\in\mathbb{Z};|j|\ge m} \frac{\gamma_j^2}{j^2} \ge K\right] \le \exp\big(m - Km^2/4\big) \qquad (m\in\mathbb{N}, K>0), \tag{3.4}$$

which also follows from Chebyshev's inequality and independence.

Let $\mu_\lambda(du) = \zeta(\lambda)^{-1}\exp(\lambda V(u))W(du)$ where

$$\zeta(\lambda) = \int_{B_K} \exp(\lambda V(u))W(du)$$

so that μ_λ is a probability measure; we can take $V(u) = \int_{\mathbb{T}} u(\theta)^4 d\theta/(2\pi)$ and W to be Brownian loop measure. Here μ_λ is Gibbs measure (1.5) with the inverse temperature β, but we prefer to work with $\lambda = -\beta > 0$ so that the convexity statements are easier to interpret.

Theorem III.1. *Under the family of Gibbs measures (1.5) associated with NLS (1.3), the random variable $u \mapsto \langle u^{\otimes k} \mid T \mid u^{\otimes k}\rangle$ with $u \in (B_K, L^2, \mu_\lambda)$ and $T \in \mathscr{L}(H^{s\otimes k})$ satisfies a Gaussian concentration of measure (3.6), the mean is a Lipschitz continuous function of β, and the mean for $\beta = 0$ is a sum over partitions of $2k$ over even decompositions.*

The statements in this theorem will be proved in this section. They involve the integral

$$G_\lambda^{(k)} = \int_{B_K} \mid u^{\otimes k}\rangle\langle u^{\otimes k} \mid \mu_\lambda(du) \tag{3.5}$$

where μ_λ is the Gibbs measure for NLS. In the defocussing case, the k-particle density matrix of an interacting quantum system with suitable initial conditions converges to its classical analogue see Ref.[1] (2.16) for the 1D case and Ref.[33] for 2D and 3D.

We can write $u = P + iQ$ for real variables (P, Q) and interpret $\langle u^{\otimes k} \mid T \mid u^{\otimes k}\rangle$ as a homogeneous polynomial in (p, q) of total degree $2k$. The following result gives concentration of measure for Lipschitz functions on (B_K, L^2, μ_λ), and shows that k-point matrices are concentrated near to their mean value.

Proposition III.2. *For $T \in \mathscr{L}(H^{s\otimes k})$ with operator norm $\|T\|$, let $g_T : B_K \to \mathbb{C}$ by $g_T(u) = \langle u^{\otimes k} \mid T \mid u^{\otimes k}\rangle$. Then there exists $\alpha = \alpha(\beta, K) > 0$ such that*

$$\mu_\lambda\Big(\big\{u \in B_K : |g_T(u) - \mathrm{trace}(G_\lambda^{(k)}T)| > r\big\}\Big) \le 4\exp\Big(\frac{-\alpha r^2}{32k^2K^{2k-1}\|T\|^2}\Big) \qquad (r>0). \tag{3.6}$$

Proof. Here g_T has mean value

$$\int_{B_K} g_T(u)\mu_\lambda(du) = \int_{B_K} \langle u^{\otimes k} \mid T \mid u^{\otimes k}\rangle \mu_\lambda(du) = \mathrm{trace}(G_\lambda^{(k)}T). \tag{3.7}$$

Also g_T is Lipschitz, with

$$|g_T(u) - g_T(v)| \le \|T\| \sum_{j=0}^{2k-1} \|u\|^j \|v\|^{2k-j-1}\|u-v\| \le 2kK^{k-1/2}\|T\|\|u-v\| \qquad (u,v\in B_K). \tag{3.8}$$

By the logarithmic Sobolev inequality Theorem 1.2(iv) of[9], there exists $\alpha = \alpha(K,\beta) > 0$ such that

$$\int_{B_K} f(u)^2\log f(u)^2\mu_\lambda(du) \le \int_{B_K} f(u)^2\mu_\lambda(du)\log\Big(\int_{B_K} f(u)^2\mu_\lambda(du)\Big) + \frac{2}{\alpha}\int_{B_K}\|\nabla f\|^2\mu_\lambda(du) \tag{3.9}$$

for all continuously differentiable $f : B_K \to \mathbb{R}$, where ∇ refers to the Fréchet derivative. In particular, we choose

$$f(u) = \exp\Big(r\,\mathrm{Re}\big(g_T(u) - \mathrm{trace}(G_t^{(k)}T)\big)/2\Big)$$

and we deduce that the moment generating function

$$\varphi(r) = \int_{B_K} \exp\Big(r\,\mathrm{Re}\big(g_T(u) - \mathrm{trace}(G_\lambda^{(k)}T)\big)\Big)\mu_\lambda(du) \tag{3.10}$$

satisfies $\varphi(0) = 1$,

$$\varphi'(0) = \int_{B_K} \mathrm{Re}\big(g_T(u) - \mathrm{trace}(G_\lambda^{(k)}T)\big)\mu_\lambda(du) = 0,$$

hence $r^{-1}\log\varphi(r) \to 0$ as $r \to 0+$. The differential inequality

$$r\varphi'(r) \le \varphi(r)\log\varphi(r) + \frac{8k^2K^{2k-1}\|T\|^2r^2}{\alpha}\varphi(r) \tag{3.11}$$

follows directly from (3.9), hence

$$\frac{d}{dr}\Big(\frac{\log\varphi(r)}{r}\Big) \le \frac{8k^2K^{2k-1}\|T\|^2}{\alpha},$$

so we obtain the concentration inequality

$$\varphi(r) \le \exp\Big(\frac{8k^2K^{2k-1}\|T\|^2r^2}{\alpha}\Big) \qquad (r \ge 0). \tag{3.12}$$

One can conclude the proof by a standard application of Chebyshev's inequality to the integral for φ in (3.10). □

To make full use of the previous result, one needs to know the mean $\mathrm{trace}(G_\lambda^{(k)}T)$ as in (3.7), which depends upon the measure in (3.5). The following shows how the mean can vary with the inverse temperature $\beta = -\lambda$.

Proposition III.3. *For $g : B_K \to \mathbb{R}$ an L-Lipschitz function, the mean values of g with respect to the measures μ_λ satisfy*

$$\Big(\int_{B_K} g(u)\big(\mu_b(du) - \mu_a(du)\big)\Big)^2 \le \frac{L^2(b-a)^2}{2\alpha}\iint_{B_K\times B_K}(V(u) - V(w))^2\mu_\lambda(du)\mu_\lambda(dw) \tag{3.13}$$

where α is the constant in (3.9) for μ_a, and some $\lambda \in (a,b)$.

Proof. We observe that $\log\zeta(\lambda)$ is a convex function of $\lambda > 0$ and by the mean value theorem, there exists $a < \lambda < b$ such that

$$\begin{aligned}\log\zeta(a) - \log\zeta(b) &= -(b-a)\frac{\zeta'(b)}{\zeta(b)} + \frac{(b-a)^2}{2}\Big(\frac{\zeta''(\lambda)}{\zeta(\lambda)} - \frac{\zeta'(\lambda)^2}{\zeta(\lambda)^2}\Big)\\ &= -(b-a)\int_{B_K} V(u)\mu_b(du) + \frac{(b-a)^2}{4}\iint_{B_K\times B_K}(V(u) - V(w))^2\mu_\lambda(du)\mu_\lambda(dw).\end{aligned} \tag{3.14}$$

Let $W_1(\mu_a,\mu_b)$ be the Wasserstein transportation distance between μ_b and μ_a for the cost function $\|u - v\|_{L^2}$, as in page 34 of Ref.[41]. Then by duality we have

$$\Big|\int_{B_K} g(u)\mu_b(du) - \int_{B_K} g(u)\mu_a(du)\Big| \le LW_1(\mu_b,\mu_a).$$

By results of Otto and Villani discussed in Ref.[41] pages 291-2, the logarithmic Sobolev inequality of Theorem 1.2(iv) Ref.[9] implies a transportation cost inequality

$$W_1(\mu_b,\mu_a) \le \Big(\frac{2}{\alpha}\mathrm{Ent}(\mu_b \mid \mu_a)\Big)^{1/2} \tag{3.15}$$

in the style of Talagrand, where the relative entropy is

$$\begin{aligned}\mathrm{Ent}(\mu_b \mid \mu_a) &= \int_{B_K} \log\frac{d\mu_b}{d\mu_a}\mu_b(da)\\ &= (b-a)\int_{B_K} V(u)\mu_b(du) - (\log\zeta(b) - \zeta(a))\\ &= \frac{(b-a)^2}{4}\iint_{B_K\times B_K}(V(u) - V(w))^2\mu_\lambda(du)\mu_\lambda(dw),\end{aligned} \tag{3.16}$$

where the final step follows from (3.14). The stated result follows on combining these inequalities. □

Proposition III.4. *The integral $G_0^{(k)}$ from (3.5) is the sum over the terms (2.6) that arise from a $\pi \in \Pi_k$ with n nonzero rows, a n-subset of $\mathbb{N}$, and an even decomposition of π into a pair $(\lambda, \rho) \in \Pi_k^2$ where λ and ρ have equal numbers of odd rows.*

Proof. The measure μ_0 is a Wiener loop measure restricted to B_k. For any sequence $(\varepsilon_n)_{n\in\mathbb{Z}} \in \{\pm 1\}^{\mathbb{Z}}$, the sequence $(\gamma_n)_{n\in\mathbb{Z}}$ with γ_n mutually independent $N(0,1)$ Gaussian random variables has the same distribution as the sequence $(\varepsilon_n\gamma)_{n\in\mathbb{Z}}$. Also, the condition $\sum_{n\in\mathbb{Z}\setminus\{0\}} \gamma_n^2/n^2 \leq K$ does not change under this transformation. Let $u_\varepsilon(\theta) = \sum_j' \varepsilon_j \zeta_j e^{ij\theta}/|j|$. We therefore have

$$G_0^{(k)} = \int_{B_K}\int_{\{\pm 1\}^{\mathbb{Z}}} \mid u_\varepsilon^{\otimes k}\rangle\langle u_\varepsilon^{\otimes k} \mid d\varepsilon\mu_0(du)$$

where $d\varepsilon$ is the Haar probability measure on the Cantor group $\{\pm 1\}^{\mathbb{Z}}$. The measure on $\{\pm\}$ is associated with tossing a fair coin, and Haar measure is the product of such probability measures. We can therefore compute the inner integral in this expression for $G_0^{(k)}$ by the same calculation that led to the corresponding statement for $J^{(k)}$, since we only used the even decomposition of partitions to derive (2.6). $\square$

We have

$$\left[\sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{\gamma_j^2}{j^2} \leq K\right] \subseteq \bigcap_{j\in\mathbb{Z}\setminus\{0\}} \left[\gamma_j^2 \leq Kj^2\right] \tag{3.17}$$

where the sets are independent under the Gaussian measure $d\mathbb{P}$, so we have a substitute for (2.7). Conditioning on the event B_K,

$$\int_{B_K} \gamma_{j_1}^{2k_1}\gamma_{j_2}^{2k_2}\dots\gamma_{j_n}^{2k_n}\mu_0(du) \leq \mathbb{P}(B_K)^{-1}\prod_{\ell=1}^{n}\int_{[\gamma^2\leq Kj_\ell^2]} \gamma^{2k_\ell}d\mathbb{P}, \tag{3.18}$$

where $\mathbb{P}(B_K)$ satisfies (3.3) and there is an approximate formula

$$\int_{[\gamma^2\leq Kj_\ell^2]} \gamma^{2k_\ell}d\mathbb{P} = \frac{(2k_\ell)!}{2^{k_\ell}k_\ell!}\exp\Big(-\Big(\frac{j_\ell^2K}{2}\Big)^{k_\ell-1/2}\frac{e^{-j_\ell^2K/2}}{\Gamma(k_\ell+1/2)}\Big). \tag{3.19}$$

IV. CONCENTRATION FOR METRIC MEASURE SPACES

In a similar spirit, we give a concentration result for k-fold stochastic integrals. This result resemble the integrability criteria of[14] which relates to a single variable. Let H_0^1 be the homogeneous Sobolev space of $v \in L^2(\mathbb{T};\mathbb{C})$ that are absolutely continuous with derivative $v' \in L^2(\mathbb{T};\mathbb{C})$ with $\int_{\mathbb{T}} v(x)dx = 0$. Let $h_j \in H_0^1$ for $j = 1,\dots,k$ be such that $\sum_{j=1}^k \int (h_j')^2dx \leq 1$, and consider $\Phi : B_K^k \mapsto \mathbb{R}^k$ given by

$$\Phi : (u_j)_{j=1}^k \mapsto \Big(\int u_j(x)h_j'(x)dx\Big)_{j=1}^k. \tag{4.1}$$

The following result describes the distribution of this $\mathbb{C}^k$-valued random variable.

Proposition IV.1. *Let ν_K be the probability measure on $\mathbb{C}^k$ that is induced from $\mu_{K,\beta}^{\otimes k}$ by Φ. Then there exists $\alpha_K > 0$ independent of k such that*

$$\int_{\mathbb{C}^k} G(w)^2\log\Big(G(w)^2/\int G^2d\nu_K\Big)\nu_K(dw) \leq \frac{2}{\alpha_K}\int_{\mathbb{C}^k}\|\nabla G(w)\|^2\nu_K(dw) \tag{4.2}$$

for all $G \in C_c^1(\mathbb{C}^k;\mathbb{R})$. The distribution ν_K has mean x_0 and satisfies

$$\int_{\mathbb{C}^k} e^{t^2\|x-x_0\|^2/2}\nu_K(dx) \leq \Big(1-\frac{t^2}{\alpha}\Big)^{-k} \qquad (t^2 < \alpha). \tag{4.3}$$

Proof. We observe that for all $u = (u_j)_{j=1}^k \in B_K^k$ and $v = (v_j)_{j=1}^k \in B_K^k$ we have

$$\|\Phi(u)-\Phi(v)\|_{\mathbb{R}^k}^2 \leq \sum_{j=1}^k\int (h_j'(x))^2dx\sum_{j=1}^k\int |u_j(x)-v_j(x)|^2dx \tag{4.4}$$

so $\Phi : (B_K^k, \ell^2(L^2)) \to (\mathbb{R}^k, \ell^2)$ is Lipschitz with constant one. Each metric probability space $(B_K, L^2, \mu_{K,\beta})$ satisfies a logarithmic Sobolev inequality with constant $\alpha_K > 0$ by Ref.[9], and the probability space $(B_K^k, \ell^2(L^2), \mu_{K,\beta}^{\otimes k})$ is a direct product of the metric probability spaces $(B_K, L^2, \mu_{K,\beta})$, hence also satisfies a logarithmic Sobolev inequality

$$\int_{B_K^k} G\circ\Phi(u)^2 \log\Big(G\circ\Phi(u)^2 \Big/ \int G\circ\Phi^2 d\mu_{K,\beta}^{\otimes k}\Big)\mu_{K,\beta}^{\otimes k}(du) \le \frac{2}{\alpha_K}\int_{B_K^k} \|\nabla G\circ\Phi(u)\|^2 \mu_{K,\beta}^{\otimes k}(du) \tag{4.5}$$

with constant α_K independent of k, by section 22 of Ref.[42].

We introduce $x_0 = \int_{\mathbb{C}^k} x\nu_K(dx)$ and consider

$$\varphi(t) = \int_{\mathbb{C}^k} e^{t\,\mathrm{Re}\langle x-x_0, y\rangle}\nu_K(dx) \tag{4.6}$$

which satisfies $\varphi(0) = 1$, $\varphi'(0) = 0$ and the differential inequality

$$t\varphi'(t) \le \varphi(t)\log\varphi(t) + \frac{t^2\|y\|^2}{2\alpha}\varphi(t) \tag{4.7}$$

follows from (4.2). This gives

$$\int_{\mathbb{C}^k} e^{t\,\mathrm{Re}\langle x-x_0, y\rangle}\nu_K(dx) \le e^{t^2\|y\|^2/(2\alpha)} \tag{4.8}$$

which we integrate against $e^{-\|y\|^2/2}$, where $y \in \mathbb{C}^k = \mathbb{R}^{2k}$, to obtain

$$\int_{\mathbb{C}^k} e^{t^2\|x-x_0\|^2/2}\nu_K(dx) \le \Big(1 - \frac{t^2}{\alpha}\Big)^{-k} \qquad (t^2 < \alpha). \tag{4.9}$$

□

The probability space $(B_K, L^2, \mu_{K,\beta})$ has a tangent space associated with infinitesimal translations. Let H^1 be the Sobolev space of $v \in L^2(\mathbb{T};\mathbb{C})$ that are absolutely continuous with derivative $v' \in L^2(\mathbb{T};\mathbb{C})$; let $H^{-1} = (H^1)^*$ be the linear topological dual space for the pairing $\langle v, w\rangle \mapsto \int_{\mathbb{T}} v(x)\overline{w(x)}dx/(2\pi)$ as interpreted via Fourier series. Then there is a Radonifying triple of continuous linear inclusions

$$H^1 \to L^2 \to (H^1)^* \tag{4.10}$$

associated with the Gibbs measure μ_K. The space H^1 has orthonormal basis $(h_n)_{n=-\infty}^{\infty} = (e^{in\theta}/\sqrt{n^2+1})_{n=-\infty}^{\infty}$ and the covariance matrix of Wiener loop is $R_0 = \mathrm{diag}[1/(1+n^2)]_{n=-\infty}^{\infty}$ with respect to this basis. By Cauchy-Schwarz, we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty} |\langle R_0 h_n, h_n\rangle| &= \int \sum_{n=-\infty}^{\infty} \Big|\int h_j du\Big|^2 \mu_{K,\beta}(du) \\ &\le \Big(\int \sum_{n=-\infty}^{\infty} (\sqrt{1+n^2})^{1+\varepsilon}\Big|\int h_n' u(x)dx\Big|^4 W(du)\Big)^{1/2}\Big(\sum_{n=-\infty}^{\infty}\int (\sqrt{1+n^2})^{-1-\varepsilon}\Big(\frac{d\mu_{K,\beta}}{dW}\Big)^2 dW\Big)^{1/2}\end{aligned} \tag{4.11}$$

for all $\varepsilon > 0$. By such simple estimates, one can deduce that there exists, for each β and $K > 0$, a self-adjoint, nonnegative and trace class operator R such that

$$\langle Rf, g\rangle_{H^1} = \int_{B_K}\int_{[0,2\pi]} f' u d\theta \int_{[0,2\pi]} \bar{g}' \bar{u} d\theta\, \mu_{K,\beta}(du), \tag{4.12}$$

which gives the covariance matrix of the Gibbs measure on H^1. This is essentially $G_{-\beta}^{(1)}$, up to the identification of Hilbert spaces in (4.10).

Cameron and Martin computed the density with respect to the Wiener measure that results from the linear translation $u \mapsto u + v$ for $v \in H^1$; their results extends to Gibbs measure with some modifications.

We momentarily suppress the dependence of functions upon time, and consider for $p, q \in H^1$, the linear transformation $P + iQ \mapsto P + p + i(Q + q)$. Cameron and Martin proved that free Wiener measure ($\beta = 0$) is mapped to a measure that absolutely continuous with respect to the free Wiener measure. Likewise, Gibbs measure is mapped to a measure absolutely continuous with respect to Gibbs measure. The total space of the tangent bundle to the $\sqrt{K}$ sphere in L^2 is

$$\{(f, h) : f \in L^2, \|f\|_{L^2}^2 = K; h \in H^1, \langle Rf, h\rangle_{H^1} = 0\}$$

which has fibres that are subspaces of H^1. With this in mind, we make a polar decomposition $P + iQ = \kappa e^{i\sigma}$ with $\kappa = \sqrt{P^2+Q^2}$ and consider $\tau = \frac{\partial\sigma}{\partial x}$.

Proposition IV.2. *For $p,q\in H^1$ the functional*

$$L(P,Q)=L(\kappa e^{i\sigma})=-\int_{\mathbb{T}}\frac{\partial p}{\partial x}\kappa\cos\sigma dx-\int_{\mathbb{T}}\frac{\partial q}{\partial x}\kappa\sin\sigma\, dx \tag{4.13}$$

is a Lipschitz functional of $P+iQ=\kappa e^{i\sigma}$ such that

$$\int_{B_K}\exp\Big(sL(P,Q)-s\int_{B_K}L(P,Q)\mu_{K,\beta}(dPdQ)\Big)\mu_{K,\beta}(dPdQ)$$
$$\le\exp\Big\{Cs^2\int_{\mathbb{T}}\Big(\Big(\frac{\partial p}{\partial x}\Big)^2+\Big(\frac{\partial q}{\partial x}\Big)^2\Big)dx\Big\}\qquad (s\in\mathbb{R}). \tag{4.14}$$

Proof. Note that $P+iQ\mapsto\kappa$ is 1-Lipschitz with

$$\big[\sigma\kappa,\kappa\nabla\sigma\big]=\frac{1}{\sqrt{P^2+Q^2}}\begin{bmatrix}P & -Q\\ Q & P\end{bmatrix}\in SO(2).$$

Also $\kappa^2\|\mathrm{Hess}\,\sigma\|$ is bounded. We have

$$L(P,Q)=\int p\Big(\frac{\partial\kappa}{\partial x}\cos\sigma-\kappa\tau\sin\sigma\Big)dx+\int q\Big(\frac{\partial\kappa}{\partial x}\sin\sigma+\kappa\tau\cos\sigma\Big)dx \tag{4.15}$$

which is bounded on L^2 with norm Λ where

$$\Lambda^2\le\int_{\mathbb{T}}\Big(\Big(\frac{\partial p}{\partial x}\Big)^2+\Big(\frac{\partial q}{\partial x}\Big)^2\Big)dx. \tag{4.16}$$

By the concentration of measure theorem for ν_K, we deduce the stated inequality. □

Definition We say that (M,d,μ) satisfies $T_2(\alpha)$ if

$$W_2(\nu,\mu)^2\le\frac{2}{\alpha}\mathrm{Ent}(\nu\mid\mu) \tag{4.17}$$

for all probability measures ν that are of finite relative entropy with respect to μ. The notation credits Talagrand, who developed the theory of such transportation inequalities. Otto and Villani showed that $LSI(\alpha)$ implies $T_2(\alpha)$; see Refs.[41] and[42]

Theorem IV.3. *Let (M,d,μ) be a metric probability space that satisfies $T_2(\alpha)$; let $(M^N,\ell^2(d),\mu^{\otimes N})$ be the direct product metric probability space. Let $L_N^\xi=N^{-1}\sum_{j=1}^N\delta_{\xi_j}$ be the empirical distribution for $\xi=(\xi_j)_{j=1}^N\in M^N$ where ξ_j distributed as μ. Then the concentration inequality holds*

$$\mu^{\otimes N}\Big(\Big\{\xi:\in M^N:\big|W_p(L_N^\xi,\mu)-\mathbb{E}W_p(L_N,\mu)\big|>\varepsilon\Big\}\Big)\le 2e^{-N\alpha\varepsilon^2/2}\qquad(\varepsilon>0) \tag{4.18}$$

for $p=1,2$.

Proof. The map between metric spaces

$$L_N:(M^N,\ell^2(d))\to(\mathrm{Prob}(M),W_2):\quad (x_j)_{j=1}^N\mapsto\frac{1}{N}\sum_{j=1}^N\delta_{x_j} \tag{4.19}$$

associated with the empirical distribution is $1/\sqrt{N}$-Lipschitz. Let $(x_j)_{j=1}^N,(y_j)_{j=1}^N\in M^N$ and consider the probability measure on $M\times M$ given by

$$\pi=\frac{1}{N}\sum_{j=1}^N\delta_{(x_j,y_j)} \tag{4.20}$$

which has marginals $L_N^x=N^{-1}\sum_{j=1}^N\delta_{x_j}$ and $L_N^y=N^{-1}\sum_{j=1}^N\delta_{y_j}$, hence gives a transport plan with cost

$$W_2(L_N^x,L_N^y)^2\le\iint_{M\times M}d(x,y)^2\pi(dxdy)=\frac{1}{N}\sum_{j=1}^N d(x_j,y_j)^2. \tag{4.21}$$

Suppose that (M,d,μ) satisfies $T_2(\alpha)$. Then we take N independent samples $\xi_1,\dots,\xi_N$, each distributed as μ so they have joint distribution $\mu^{\otimes N}$ on $(M^N,\ell^2(d))$, where by independence[8] Theorem 1.2, $(M^N,\ell^2(d),\mu^{\otimes N})$ also satisfies $T_2(\alpha)$. By forming the empirical distribution, we obtain a map $L_N:(M^N,\ell^2(d))\to(\mathrm{Prob}M,W_2)$. Then $\varphi(\xi)=\sqrt{N}W_p(L_N^\xi,\mu)$ is 1-Lipschitz $(M^N,\ell^2(d))\to\mathbb{R}$, since by the triangle inequality and (4.21),

$$|\varphi(\xi)-\varphi(\eta)|\le\sqrt{N}W_p(L_N^\xi,L_N^\eta)\le\sqrt{N}W_2(L_N^\xi,L_N^\eta)\le\sum_{j=1}^N d(\xi_j,\eta_j)^2; \tag{4.22}$$

hence φ satisfies the concentration inequality

$$\int_{M^N}\exp\Big(t\varphi(\xi)-t\int\varphi d\mu^{\otimes N}\Big)d\mu^{\otimes N}\le\exp(t^2/(2\alpha))\qquad(t\in\mathbb{R}). \tag{4.23}$$

Then the stated concentration inequality follows from Chebyshev's inequality.

□

Theorem IV.3 gives a metric version of Sanov's theorem on the empirical distribution; see page 70 of[18]. There are related results in Bolley's thesis[10]. By[16], Theorem IV.3 applies to Haar probability measure on $SO(3)$ and normalized area measure on $\mathbb{S}^2$, as is relevant in section 7 below. However, to ensure that $\mathbb{E}W_2(L_N,\mu)\to0$ as $N\to\infty$, it is convenient to reduce to one-dimensional distributions, where we use the following integral formula. For distributions μ and ν on $\mathbb{R}$ with cumulative distribution functions F and G, we write $W_p(\mu,\nu)=W_p(F,G)$.

Proposition IV.4. *Let ξ_1 be a real random variable with finite fourth moment, and cumulative distribution function F. Let $\xi_1,\dots,\xi_N$ be mutually independent copies of ξ_1 giving an empirical measure $L_N^\xi=N^{-1}\sum_{j=1}^N\delta_{\xi_j}$ with cumulative distribution function $F_N^\xi(t)$. Then*

$$\int_{\mathbb{R}^N}W_1(F_N^\xi,F)\mu^{\otimes N}(d\xi)\le\frac{1}{\sqrt{N}}\int_{-\infty}^\infty\sqrt{F(t)(1-F(t))}dt. \tag{4.24}$$

Proof. Let H be Heaviside's unit step function; then $F_N^\xi(t)=N^{-1}\sum_{j=1}^N H(t-\xi_j)$, so $\sqrt{N}(F_N(t)-F(t))$ is a sum of mutually independent and bounded random variables with mean zero. Also, as in the weak law of large numbers, we have

$$\mathbb{E}W_1(F_N,F)=\int_{-\infty}^\infty\mathbb{E}|F_N(t)-F(t)|\,dt\le\int_{-\infty}^\infty\Big(\mathbb{E}\big((F_N(t)-F(t))^2\big)\Big)^{1/2}dt=\frac{1}{\sqrt{N}}\int_{-\infty}^\infty\sqrt{F(t)(1-F(t))}\,dt,$$

where the integral is finite by Chebyshev's inequality since $\mathbb{E}\xi^4$ is finite. Compare with the result of[11], which are effective for very large N.

□

Proposition IV.5. *Suppose that ξ_1 has distribution μ on $\mathbb{S}^2$ where μ is absolutely continuous with respect to the normalized area measure ν_1, and $d\mu=fd\nu_1$ where f is bounded with $\|f\|_\infty\le M$. Let ξ_j be mutually independent copies of ξ_1, and let L_N^ξ be the empirical measure from N samples. Then*

$$\int_{(\mathbb{S}^2)^N}W_1(L_N^\xi,\mu)d\mu^{\otimes N}=O(N^{-1/4})\qquad(N\to\infty).$$

Proof. Let $g:\mathbb{S}^2\to\mathbb{R}$ be 1-Lipschitz, and suppose without loss of generality that g has $\int_{\mathbb{S}^2}g(x)\nu_1(dx)=0$; then g is bounded with $\|g\|_\infty\le\pi$. Given $\delta>0$, by considering squares for coordinates in longitude and colatitude, we choose disjoint and connected subsets E_ℓ with diameter $\mathrm{diam}(E_\ell)\le\delta$ and $\nu_1(E_\ell)\le\delta^2$ and $\mu(E_\ell)\le M\nu_1(E)$ such that $\cup_\ell E_\ell=\mathbb{S}^2$. We can arrange that there are S_δ such sets E_ℓ, where $S_\delta\le C/\delta^2$. Let $\mathscr{F}$ be the σ-algebra that is generated by the E_ℓ, take conditional expectations in $L^2(\nu_1)$, and observe that

$$\begin{aligned}\int_{\mathbb{S}^2}g(x)dL_N^\xi-\int_{\mathbb{S}^2}g(x)d\mu(x)&=\int_{\mathbb{S}^2}\big(g(x)-\mathbb{E}g\mid\mathscr{F})\big)dL_N^\xi+\int_{\mathbb{S}^2}\big(\mathbb{E}(g\mid\mathscr{F})-g(x)\big)d\mu(x)\\&\quad+\int_{\mathbb{S}^2}\mathbb{E}(g\mid\mathscr{F})(x)\big(dL_N^\xi(x)-d\mu(x)\big)\end{aligned} \tag{4.25}$$

where we have bounds

$$\Big|\int_{\mathbb{S}^2}\big(g(x)-\mathbb{E}(g\mid\mathscr{F})\big)dL_N^\xi\Big|\le\sup_\ell\mathrm{Lip}(g)\mathrm{diam}(E_\ell)\le\delta, \tag{4.26}$$

$$\Big|\int_{\mathbb{S}^2}\big(g(x)-\mathbb{E}(g\mid\mathscr{F})\big)d\mu\Big|\le\sup_\ell\mathrm{Lip}(g)\mathrm{diam}(E_\ell)\le\delta, \tag{4.27}$$

and the identity

$$\int_{\mathbb{S}^2} \mathbb{E}(g \mid \mathscr{F})(x)\big(dL_N^{\xi}(x) - d\mu(x)\big) = \sum_{\ell} \frac{\int_{E_\ell} g(x)d\nu_1(x)}{\nu_1(E_\ell)} \int_{E_\ell} (dL_N^{\xi} - d\mu),$$

so by Cauchy–Schwarz, we have

$$\int_{(\mathbb{S}^2)^N} \Big(\int_{\mathbb{S}^2} \mathbb{E}(g \mid \mathscr{F})(x)\big(dL_N^{\xi}(x) - d\mu(x)\big)\Big)^2 d\mu^{\otimes N}$$
$$\leq \sum_{\ell} \nu_1(E_\ell)\|g\|_\infty^2 \sum_{\ell} \frac{1}{\nu_1(E_\ell)} \int_{(\mathbb{S}^2)^N} (L_N^{\xi}(E_\ell) - \mu(E_\ell))^2 d\mu^{\otimes N} \tag{4.28}$$

where $L_N^{\xi}(E_\ell) - \mu(E_\ell) = N^{-1}\sum_{j=1}^N (\mathbb{I}_{E_\ell}(\xi_j) - \mu(E_\ell))$ is a sum of independent random variables with mean zero and variance $N^{-1}\mu(E_\ell)(1-\mu(E_\ell))$, so

$$\int_{(\mathbb{S}^2)^N} \Big(\int_{\mathbb{S}^2} \mathbb{E}(g \mid \mathscr{F})(x)\big(dL_N^{\xi}(x) - d\mu(x)\big)\Big)^2 d\mu^{\otimes N} \leq \frac{1}{N}\|g\|_\infty^2 \sum_{\ell} M \leq \frac{\pi^2 CM}{\delta^2 N}. \tag{4.29}$$

Choosing $\delta = N^{-1/4}$ we make both (4.26) and (4.29) small, which gives the stated result.

□

Remark Consider the discrete metric δ on $[0,1]$, and observe that $\mathbb{I}_A$ gives a 1-Lipschitz function on $[0,1]$ for all open $A \subseteq [0,1]$. Then we have $\int \mathbb{I}_A(x)(d\mu(x) - d\nu(x)) = \mu(A) - \nu(A)$, so by maximizing over A we obtain the total variation norm $\|\mu - \nu\|_{var}$. With μ a continuous measure and ν a purely discrete measure, such as an empirical measure, we have $\|\mu - \nu\|_{var} = 1$. The Propositions IV.4 and IV.5 depend upon the choice of cost function as well as the measures.

The Gibbs measure (1.5) was defined using random Fourier series. This construction gives us a sequence of finite-dimensional probability spaces which approximate the space $(B_K, L^2, \mu_{K,\beta})$. To make this idea precise, we recall some definitions from Ref.[40].

Definition IV.6. (Convergence of metric measure spaces)

(i) For M a nonempty set, a pseudometric is a function $\delta : M \to [0,\infty]$ such that

$$\delta(x,y) = \delta(y,x), \quad \delta(x,x) = 0, \quad \delta(x,z) \leq \delta(x,y) + \delta(y,z) \qquad (x,y,z \in M); \tag{4.30}$$

then (M,δ) is a pseudometric space.

(ii) Given pseudo metric spaces (M_1,δ_1) and (M,δ_2), a coupling is a pseudo metric $\delta : M \to [0,\infty]$ where $M = M_1 \sqcup M_2$ such that $\delta \mid M_1 \times M_1 = \delta_1$ and $\delta \mid M_2 \times M_2 = \delta_2$.

(iii) Suppose that $\hat{M}_1 = (M_1,\delta_1,\mu_1)$ and $\hat{M}_2 = (M_2,\delta_2,\mu_2)$ are complete separable metric spaces endowed with probability measures. Consider a coupling (M,δ) and a probability measure π on $M_1 \times M_2$ with marginals $\pi_1 = \mu_1$ and $\pi_2 = \mu_2$. Then the L^2 distance between $\hat{M}_1$ and $\hat{M}_2$ is

$$\mathscr{D}_{L^2}(\hat{M}_1,\hat{M}_2) = \inf_{\delta,\pi}\Big(\int_{M\times M} \delta(x,y)^2 \pi(dxdy)\Big)^{1/2} \tag{4.31}$$

Let D_n be the Dirichlet projection taking $\sum_{k=-\infty}^{\infty}(a_k + ib_k)e^{ik\theta}$ to $\sum_{k=-n}^{n}(a_k + ib_k)e^{ik\theta}$. Following[13], we truncate the random Fourier series of $u = P + iQ = \sum_{k=-\infty}^{\infty}(a_k + ib_k)e^{ik\theta}$ to $u_n = P_n + iQ_n = \sum_{k=-n}^{n}(a_k + ib_k)e^{ik\theta}$ and correspondingly modify the Hamiltonian to

$$H_3^{(n)}((a_k),(b_k)) = \frac{1}{2}\sum_{k=-n}^{n} k^2(a_k^2 + b_k^2) + \frac{\beta}{4}\int\Big|\sum_{k=-n}^{n}(a_k + ib_k)e^{ik\theta}\Big|^4 \frac{d\theta}{2\pi} \tag{4.32}$$

for the real canonical variables $((a_k,b_k))_{k=-n}^{n}$. Then the canonical equations become a coupled system of ordinary differential differential equations in the Fourier coefficients. We introduce the polar decomposition $P_n + iQ_n = \kappa_n e^{i\sigma_n}$, and observe that in terms of these noncanonical variables, the Hamiltonians $H_1^{(n)} = \int_{\mathbb{T}} \kappa_n^2 d\theta$ and

$$H_3^{(n)} = \frac{1}{2}\int_{\mathbb{T}}\Big(\Big(\frac{\partial \kappa_n}{\partial \theta}\Big)^2 + \kappa_n^2\Big(\frac{\partial \sigma_n}{\partial \theta}\Big)^2\Big)\frac{d\theta}{2\pi} + \frac{\beta}{4}\int_{\mathbb{T}} \kappa_n^4 \frac{d\theta}{2\pi} \tag{4.33}$$

are invariants under the flow.

The corresponding Gibbs measure is

$$d\mu^{(n)}_{K,\beta} = Z(K,\beta,n)^{-1}\mathbb{I}_{B_K}(u_n)\exp\Big(\frac{-\beta}{4}\int_{\mathbb{T}}|u_n(\theta)|^4\frac{d\theta}{2\pi}\Big)W(du_n) \tag{4.34}$$

in which $W(du_n)$ is the finite dimensional projection of Wiener loop measure and is defined in terms of the Fourier modes as

$$W(du_n) = \prod_{j=-n;j\neq 0}^{n}\exp\Big(-\frac{j^2}{2}(a_j^2+b_j^2)\Big)\frac{j^2 da_j db_j}{2\pi}. \tag{4.35}$$

Consider the map $u(x,t)\mapsto u(x+h,t)$ of translation in the space variable. This commutes with D_n, and the Gibbs measures $\mu^{(n)}_{K,\beta}$ are all invariant under this translation. In terms of Fourier components, we have $M_\infty = B_K$ and

$$M_n = \Big\{(a_j,b_j)_{j=-n}^n : a_j,b_j\in\mathbb{R} : \sum_{j=-n}^{n}(a_j^2+b_j^2)\leq K\Big\} \tag{4.36}$$

with the canonical inclusions of metric spaces $(M_1,\ell^2)\subset(M_2,\ell^2)\subset\cdots\subset(M_\infty,\ell^2)$ defined by adding zeros at the start and end of the sequences, which gives a sequence of isometric embeddings for the ℓ^2 metric on sequences. When we identify $(a_j,b_j)_{j=-n}^n$ with $\sum_{j=-n}^n(a_j+ib_j)e^{ij\theta}$, then we have a corresponding embedding for the L^2 metric.

Here $(M_n,L^2,\mu^{(n)}_{K,\beta})$ is a finite-dimensional manifold and a metric probability space. We now show that these spaces converge to $(M_\infty,L^2,\mu_{K,\beta})$ as $n\to\infty$.

Lemma IV.7. *(i) Suppose that $0<-\beta K<3/(14\pi^2)$. Then $\hat{M}_n=(M_n,L^2,\mu^{(n)}_{K,\beta})$ has*

$$\mathscr{D}_{L^2}(\hat{M}_n,\hat{M}_\infty)\to 0 \qquad (n\to\infty). \tag{4.37}$$

(ii) The measures $\mu^{(n)}_{K,\beta}$ converge in total variation norm to $\mu_{K,\beta}$ as $n\to\infty$.

Proof. (i) This is proved in Theorem 3.2 of Ref.[7]; see also Example 3.8 of Ref.[40]. Let $W_2(\mu^{(n)},\mu)$ be the Wasserstein transportation distance between free Brownian loop measure μ and the pushforward of μ under the Dirichlet projection, $\mu^{(n)}=D_n\sharp\mu$, for the cost function $\|u-v\|^2_{L^2}$.

The key point is

$$\begin{aligned} W_2(\mu^{(n)},\mu)^2 &\leq \int \|D_n u-u\|^2_{L^2}\mu(du)\\ &=\mathbb{E}\sum_{k:|k|>n}\frac{|\gamma_k|^2}{k^2}=O\Big(\frac{1}{n}\Big) \qquad (n\to\infty). \end{aligned} \tag{4.38}$$

(ii) The measures $\mu^{(n)}_{K,\beta}$ converge in total variation norm to μ_K, by an observation of McKean[36] in his step 7. By M. Riesz's theorem, there exists $c_4>0$ such that $\int_{\mathbb{T}}|D_nu|^4d\theta\leq c_4\int_{\mathbb{T}}|u|^4d\theta$, and by[31] the integral

$$\int_{B_K}\exp\Big(\lambda c_4\int_{\mathbb{T}}|u(\theta)|^4d\theta\Big)W(du) \tag{4.39}$$

is finite, so we can use the integrand as a dominating function to show

$$\int_{B_K}\Big|\exp\Big(\lambda\int_{\mathbb{T}}|D_nu(\theta)|^4d\theta\Big)-\exp\Big(\lambda\int_{\mathbb{T}}|u(\theta)|^4d\theta\Big)\Big|W(du)\to 0 \qquad (n\to\infty). \tag{4.40}$$

□

Proposition IV.8. *Let $(M_n\sqcup M_\infty,\delta_n)$ be a coupling of (M_n,L^2) and $M_\infty,L^2)$, and let $\varphi:(M_n\sqcup M_\infty,\delta_n)\to\mathbb{R}$ be a Lipschitz function. Then*

$$\int_{M_n}\varphi(u_n)\mu^{(n)}_{K,\beta}(du_n)\to\int_{M_\infty}\varphi(u)\mu_{K,\beta}(du) \qquad (n\to\infty). \tag{4.41}$$

Proof. We can introduce a pseudo metric δ_n on $M_n \cup M_\infty$ that restricts to the L^2 metric on M_n and M_∞, and apply (4.42) to Lipschitz functions $\varphi : (M_n \sqcup M_\infty, \delta_n) \to \mathbb{R}$. We can regard $M_n \times M_\infty$ as a subset of $M \times M = (M_n \sqcup M_\infty) \times (M_n \sqcup M_\infty)$. Note that for a Lipschitz function $\varphi : M \to \mathbb{R}$ such that $|\varphi(x) - \varphi(y)| \le \delta(x,y)$ for all $x, y \in M$, we have

$$\begin{aligned}\int_{M_n} \varphi(u_n)\mu^{(n)}_{K,\beta}(du_n) - \int_{M_\infty} \varphi(u)\mu_{K,\beta}(du) &= \iint_{M_n\times M_\infty} (\varphi(u_n)-\varphi(u))\pi(dudu_n)\\ &\le \iint_{M_n\times M_\infty} \delta(u_n,u)\pi(du_ndu)\\ &\le \Big(\iint_{M_n\times M_\infty} \delta(u_n,u)^2\pi(du_ndu)\Big)^{1/2}\\ &= \mathscr{D}_{L^2}(\hat{M}_n,\hat{M}_\infty). \end{aligned} \tag{4.42}$$

□

For example, with $u = \sum_{n=-\infty}^{\infty}(a_k + ib_k)e^{ik\theta}$ we introduce $D_n u = \sum_{k=-n}^{n}(a_k + ib_k)e^{ik\theta}$; then $\varphi(u) = \|D_n u\|_{L^2}$ and $\psi(u) = \|u - D_n u\|_{L^2}$ give Lipschitz functions $\varphi, \psi : (B_K, L^2) \to \mathbb{R}$.

Proposition IV.9. *For $0 < \gamma < 1/16$ and fixed $0 < t < t_0$, the map $x \mapsto u(x,t) \in L^4$ is γ- Hölder continuous, so that* $\sup\{\|u(x+h,t) - u(x,t)\|_{L^4_x}/|h|^\gamma); h \ne 0\}$ *is almost surely finite.*

Proof. We prove that for $0 < t < t_0$, we have $C = C(t_0)$ such that

$$\int_{B_K} \big\|u(x+h,t) - u(x,t)\big\|^{16}_{L^4_x}\mu_{K,\beta}(du) \le Ch^2 \tag{4.43}$$

so $x \mapsto u(x,t) \in L^4_x$ is γ- Hölder continuous for $0 < \gamma < 1/16$ by the Kolmogorov–Čentsov theorem, as Ref.[28]. To obtain (4.43), let $J_{3/8}(x) = \sum' e^{ikx}/|k|^{3/8}$ so that $J_{3/8}(x)|x|^{5/8}$ is bounded on $(-\pi,\pi)$ and $J_{3/8} \in L^{4/3}(-\pi,\pi)$. Then by Young's inequality for convolutions, with $|D| : e^{inx} \mapsto |n|e^{inx}$ we have

$$\big\|u(x+h,t) - u(x,t)\big\|_{L^4_x} \le \big\|J_{3/8}\big\|_{L^{4/3}}\big\||D|^{3/8}u(x+h,t) - |D|^{3/8}u(x,t)\big\|_{L^2_x} \tag{4.44}$$

Then by Bourgain's estimate on the solutions of NLS from Ref.[6], there exists $C(t_0)$ such that

$$\big\|u(x+h,t) - u(x,t)\big\|_{H^{3/8}_x} \le C(t_0)\big\|u(x+h,t) - u(x,t)\big\|_{H^{3/8}_x} \tag{4.45}$$

where

$$\begin{aligned}&\int_{B_K} \big\|u(x+h,t) - u(x,t)\big\|^{16}_{H^{3/8}_x}\mu_{K,\beta}(du)\\ &\quad\le \Big(\int_{B_K} \big\|u(x+h,t) - u(x,t)\big\|^{32}_{H^{3/8}_x}W(du)\Big)^{1/2}\Big(\int_{B_K}\Big(\frac{d\mu_{K,\beta}}{dW}\Big)^2 W(du)\Big)^{1/2}.\end{aligned} \tag{4.46}$$

By basic results about Gaussian series, the first factor on the right-hand side is bounded by the eighth power of

$$\sum_{k=1}^{\infty}\frac{k^{3/4}(1-\cos hk)}{k^2} \le Ch^{1/4}, \tag{4.47}$$

so we obtain (4.43). Also, by rotation invariance of the Gibbs measure, we have

$$\int_{B_K} \big|u(\theta+h,t) - u(\theta,t)\big|^4\mu_{K,\beta}(du) = \int_{B_K}\big\|u(x+h,t) - u(x,t)\big\|^4_{L^4_x}\mu_{K,\beta}(du),$$

which is

$$\begin{aligned}&\le \Big(\int_{B_K}\big\|u(x+h,0) - u(x,0)\big\|^8_{L^4_x}W(du)\Big)^{1/2}\Big(\int_{B_K}\Big(\frac{d\mu}{dW}\Big)^2 W(du)\Big)^{1/2}\\ &\le C\Big(\sum_{k=1}^{\infty}\frac{1-\cos hk}{k^2}\Big)^2 \le Ch^2,\end{aligned}$$

so $x \mapsto u(x,t)$ is $1/4$-Hölder continuous along solutions in the support of the Gibbs measure. □

V. HASIMOTO TRANSFORM

We recall the Hasimoto[25] transform, which associates with a solution $u \in C^2$ of (1.3) a space curve in $\mathbb{R}^3$ with moving frame $\{T,N,B\}$; Hasimoto considered the case $\beta = -1/2$. In the present context, u is associated with the space derivative of a tangent vector T to a unit speed space curve, so the curvature is $\kappa = \|\frac{\partial T}{\partial x}\|$. We have a polar decomposition $u = \kappa e^{i\sigma}$ where $\sigma(x,t) = \int_0^x \tau(y,t)dy$ and τ is the torsion. Then the Serret–Frenet formula is

$$\frac{\partial}{\partial x}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{5.1}$$

so the frame develops along the space curve. Let $X = [T;N;B] \in SO(3)$, and $\Omega_1(x,t)$ the matrix in (5.1). When $\Omega_1(\cdot,t) \in C(\mathbb{T};so(3))$, the solution $X(\cdot,t) \in C([0,2\pi];SO(3))$ to (5.1) is 2π periodic up to a multiplicative monodromy factor $U(t) \in SO(3)$ such that $X(x+2\pi,t) = X(x,t)U(t)$.

The frame also evolves with respect to time, so that with $\mu = -\frac{\partial\sigma}{\partial t} - \beta\kappa^2$, we have

$$\frac{\partial}{\partial t}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & -\tau\kappa & \frac{\partial\kappa}{\partial x} \\ \tau\kappa & 0 & -\mu \\ -\frac{\partial\kappa}{\partial x} & \mu & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{5.2}$$

Let Ω_2 denote the matrix in Equation (5.2). For a pair of coupled ODE $dX/dx - \Omega_1 X = 0$ and $dX/dt - \Omega_2 X = 0$, the corresponding Lax pair is

$$\frac{\partial\Omega_1}{\partial t} - \frac{\partial\Omega_2}{\partial x} + \big[\Omega_1, \Omega_2\big] = 0.$$

Lemma V.1. *(Hasimoto) If u is a C^2 function that satisfies the nonlinear Schrödinger equation, then the coupled pair of differential equations is consistent in the sense that there exists a local solution of the pair of ODE, and there exists a local solution of Lax pair.*

Thus the frame $X \in SO(3)$ evolves along the solution $P + iQ \in B_K$ of NLS, and we can regard $d/dx - \Omega_1$ and $d/dt - \Omega_2$ as connections for this evolution. Both of the coefficient matrices are real and skew symmetric. One can check that a solution of the integral equation

$$X(x,t) = X_0(x) + t\Omega_2(0,0)X_0(0) + \int_0^x \int_0^t \Big(\frac{\partial\Omega_1(y,s)}{\partial t} + \Omega_1(y,s)\Omega_2(y,s)\Big)X(y,s)dsdy \tag{5.3}$$

satisfies

$$X(x,0) = X_0(x), \qquad \frac{\partial X(x,0)}{\partial t} = \Omega_2(x,0)X_0(x),$$

$$\frac{\partial^2 X(x,t)}{\partial x\partial t} = \Big(\frac{\partial\Omega_1(x,t)}{\partial t} + \Omega_1(x,t)\Omega_2(x,t)\Big)X(x,t),$$

so smooth solutions are given in terms of an integral equation.

From the Serret–Frenet formulas the components of the acceleration along the space curve satisfy

$$\begin{aligned} \Big\|T \times \frac{\partial^2 T}{\partial x^2}\Big\|^2 &= \Big(\frac{\partial\kappa}{\partial x}\Big)^2 + \kappa^2\tau^2 = \Big(\frac{\partial Q}{\partial x}\Big)^2 + \Big(\frac{\partial P}{\partial x}\Big)^2, \\ \Big(T \cdot \frac{\partial^2 T}{\partial x^2}\Big)^2 &= \kappa^4 = \big(P^2 + Q^2\big)^2. \end{aligned} \tag{5.4}$$

The total curvature of the space curve is

$$\int_{\mathbb{T}} \kappa(x)^2 dx = \int_{\mathbb{T}} (P^2 + Q^2)dx = H_1(P,Q), \tag{5.5}$$

which is an invariant under the flow associated with the NLS.

Proposition V.2. *Let*

$$H_2(P,Q) = -\int_{\mathbb{T}} P(x)Q'(x)dx. \ \textit{Then,} \tag{5.6}$$

(i) $-H_2$ is convergent almost surely and is invariant under the flow associated with NLS,
(ii) $-H_2$ represents the area that is enclosed by the contour $\{u(x) : x \in [0,2\pi]\}$ in the complex plane, and

$$H_2 = \frac{1}{2}\int_{\mathbb{T}} \kappa^2 \tau dx; \tag{5.7}$$

(iii) $H_2^2 \leq 4^{-1}H_1H_3$ for $\beta > 0$, where H_1 is given in Equation (5.5) *and H_3 is given in Equation* (1.1).

Proof. (i) The invariance of H_2 was noted in Ref.[37] and can be proved by differentiating through the integral sign and using the canonical equations. We have a series

$$\int_{\mathbb{T}} \bar{u}(\theta,t)\frac{\partial u}{\partial\theta}(\theta,t)\frac{d\theta}{2\pi} = \lim_{N\to\infty}\sum_{j=-N}^{N}\overline{\hat{u}(j)}ij\hat{u}(j)$$

which converges almost surely. This follows since

$$\begin{aligned}\int_{B_K} \sup_N \Big|\sum_{j=-N}^{N}\overline{\hat{u}(j)}ij\hat{u}(j)\Big|^p \mu_{K,\beta}(du)\\ \leq \Big(\int_{B_K}\Big(\frac{d\mu_{K,\beta}}{dW}\Big)^2 dW\Big)^{1/2}\Big(\int_{B_K}\sup_N\Big|\sum_{j=-N}^{N}\overline{\hat{u}(j)}ij\hat{u}(j)\Big|^{2p}W(du)\Big)^{1/2},\end{aligned} \tag{5.8}$$

where the final integral involves the series

$$\lim_{N\to\infty}\sum_{j=-N}^{N}\overline{\hat{u}(j)}ij\hat{u}(j) = \sum_{j=1}^{\infty}\frac{|z_j|^2-|z_{-j}|^2}{j} \tag{5.9}$$

which is a martingale; by Fatou's Lemma, we have

$$\begin{aligned}\int_{L^2}\exp\Big(\lambda\sum_{j=1}^{\infty}\frac{|z_j|^2-|z_{-j}|^2}{j}\Big)dW &= \prod_{j=1}^{\infty}\int_{L^2}\exp\Big(\frac{\lambda(|z_j|^2-|z_{-j}|^2)}{j}\Big)dW\\ &= \Big(\frac{2\pi\lambda}{\sin 2\pi\lambda}\Big)^{1/2} \qquad (-1/2<\lambda<1/2),\end{aligned} \tag{5.10}$$

so the series in (5.9) is marginally exponentially integrable. Hence the integrals in (5.8) converge by the L^p martingale maximal theorem for all $1 < p < \infty$.

(ii) One can write H_2 in terms of $P + iQ = \kappa e^{i\sigma}$, and make a change of variables to obtain

$$\kappa = \frac{\partial(P,Q)}{\partial(\kappa,\sigma)}$$

and

$$H_2 = \frac{1}{2}\int_{\mathbb{T}}(P'Q - PQ')dx = \frac{1}{2}\int_{\mathbb{T}}\kappa^2\tau dx.$$

To interpret this as an area, We write $\theta \in [0,2\pi]$ for the space variable and extend functions on $[0,2\pi]$ to harmonic functions on the unit disc via the Poisson kernel. Then by Green's theorem, we can express this invariant in terms of the area of the image of $\mathbb{D}$ under the map to $P+iQ$, as in

$$-H_2 = \iint_{\mathbb{D}}\frac{\partial(P,Q)}{\partial(x,y)}dxdy. \tag{5.11}$$

This is similar to Lévy's stochastic area, as discussed in Example 5.1 of[26].

(iii) We then have

$$\Big(\int_{\mathbb{T}}\kappa^2\tau dx\Big)^2 \leq \int_{\mathbb{T}}\kappa^2 dx\int_{\mathbb{T}}\kappa^2\tau^2 dx$$

which is bounded in terms of other invariants, with $H_2^2 \leq 4^{-1}H_1H_3$.

□

Remark V.3.

(i) Bourgain[13] interprets H_2 in terms of momentum (5.70).

(ii) With M_n as in (4.36), the space $C^\infty(M_n;\mathbb{R})$ is a Poisson algebra for the bracket $\{f,g\}=\sum_{j=-n}^{n}\frac{\partial(f,g)}{\partial(a_j,b_j)}$, and the canonical equations arise with Hamiltonian $H_3^{(n)}$ on M_n. Let Q be the ring of quaternions, and extend the Poisson bracket to $C^\infty(M_n;Q)$ via $\{f\otimes X, g\otimes Y\}=\{f,g\}\otimes XY$. Then $(\mathbb{R}^3,\times)$ may be realised as $Q/\mathbb{R}I$ and $(so(3),[\cdot,\cdot])\cong(\mathbb{R}^3,\times)$; see Example 2.3 of Ref.[44]. This Lie algebra is also the Lie algebra of $SU(2)$, and there is a $2-1$ group homomorphism $SU(2)\to SO(3)$. Hence some of the following results may be expressed in terms of $SU(2)$, which is the form in which a Lax pair for the NLSE was presented, see Ref.[45] and Ref.[43] (Subsection 8.3.2).

(iii) Suppose that $T\in C^2([0,a]\times[0,b];\mathbb{S}^2)$, so that $T(x,t)$ represents the spin of the particle at (x,t) and let

$$E(T)=\int_0^a\Big\|\frac{\partial T}{\partial x}(x,t)\Big\|^2dx, \tag{5.12}$$

which corresponds to our 5.5. One can consider infinitesimal variations $T\mapsto T+T\times V$ and thereby compute $\frac{\partial E}{\partial T}$. In the focusing case $\beta=-1$, Ding[20] introduces a symplectic structure on the space of such maps such that the Hamiltonian flow is

$$\frac{\partial T}{\partial t}=T\times\frac{\partial^2 T}{\partial x^2} \tag{5.13}$$

which corresponds to Heisenberg's equation for the one-dimensional ferro-magnet, and gives the top entry of (5.2). There is a a gauge equivalence between the focussing NLS and Heisenberg's ferro-magnet. There is also a gauge equivalence between the defocussing NLS and a hyperbolic version of the ferromagnet in which the standard cross product is modified. We have

$$\Big|\frac{\partial^2 T}{\partial x^2}\Big|^2=\Big(\frac{\partial\kappa}{\partial x}\Big)^2+\kappa^2\tau^2+\kappa^4=\Big|\frac{\partial u}{\partial x}\Big|^2+|u|^4. \tag{5.14}$$

(iv) As in[21], the space

$$\mathscr{L}(SO(3))=\{g:[0,2\pi]\to SO(3);g\quad\text{continuous},\quad g(0)=g(2\pi)\}$$

with pointwise multiplication is a loop group, and its Lie algebra may be regarded as

$$H_0^1(so(3))=\Big\{h;[0,2\pi]\to so(3);h\quad\text{absolutely continuous},\quad h(0)=h(2\pi)=0,\int_0^{2\pi}\|h'(x)\|^2_{so(3)}dx<\infty\Big\}.$$

The aim of the next section is to interpret the Lax pair suitably for solutions which are typically not differentiable and for which we have a pair of stochastic differential equations with random matrix coefficients.

VI. GIBBS MEASURE TRANSPORTED TO THE FRAMES

The compact Lie group $SO(3)$ of real orthogonal matrices with determinant one is a subset of $M_{3\times3}(\mathbb{R})$, which has the scalar product $\langle X,Y\rangle=\mathrm{trace}(XY^\top)$ and associated metric $d(X,Y)=\langle X-Y,X-Y\rangle^{1/2}$ such that $\langle XU,YU\rangle=\langle X,Y\rangle$ and $d(XU,YU)=d(X,Y)$ for all $U\in SO(3)$ and $X,Y\in M_{3\times3}(\mathbb{R})$. The Lie group $SO(3)$ has tangent space at the identity element give by the skew symmetric matrices $so(3)$, so the tangent space $T_XSO(3)$ at $X\in SO(3)$ consists of $\{\Omega X:\Omega\in so(3)\}$, where $so(3)$ is a Lie algebra for $[x,y]=xy-yx$, $x,y\in so(3)$, and the exponential map is surjective $so(3)\to SO(3)$.

Consider the differential equation

$$\frac{dX}{dt}=\Omega(t)X;\quad X(0)=X_0 \tag{6.1}$$

where $t\in[0,1]$ is the evolving time, and $X\in SO(3)$. We consider a column vector $x\in\mathbb{R}^3$, satisfying $\frac{dx}{dt}=\Omega x$ which gives a velocity, and $\|x\|=1$ because $\Omega\in so(3)$. Following Otto's interpretation[41] of optimal transport in the setting of partial differential equations, one constructs a weakly continuous family of probability measures, $\tilde{\nu}_t$ on $\mathbb{S}^2$ for $t \in [0,1]$, which satisfy the weak continuity equation,

$$\frac{\partial \tilde{\nu}_t}{\partial t} + \nabla \cdot \left(\Omega x \tilde{\nu}_t\right) = 0. \tag{6.2}$$

Likewise the differential equation (6.1) gives a weakly continuous family of probability measures, ν_t on $SO(3)$. If the integral

$$\int_0^1 \int_{SO(3)} \|\Omega X\|^2_{M_{3\times 3}(\mathbb{R})} \nu_t(dX)dt < \infty, \tag{6.3}$$

and ΩX is locally bounded, then ΩX is locally Lipschitz and ν_t is the unique solution to the weak continuity equation by Thm 5.34 of Ref.[41]. Recall that for the operator norm on $M_{3\times 3}(\mathbb{R})$, $\|A\| = \sup\{\|Ay\| : y \in \mathbb{R}^3\}$, where $\|X\| = 1$ for all $X \in SO(3)$ so $\|\Omega X\| \leq \|\Omega\|$.

The weak continuity equation is equivalent to

$$\int_{SO(3)} f(X)\nu_t(dX) = \int_{SO(3)} f(X_t(X_0))\nu_0(dX_0) \tag{6.4}$$

for all $f \in C(SO(3);\mathbb{R})$, where $X_0 \mapsto X_t(X_0)$ gives the dependence of the solution of (6.1) on the initial condition. The velocity field ΩX is associated with a transportation plan taking ν_{t_1} to ν_{t_2} which is possibly not optimal, but does give an upper bound on the Wasserstein distance for the cost $d(X,Y)^2$ on $SO(3)$ of

$$\frac{W_2(\nu_{t_2},\nu_{t_1})^2}{t_t - t_1} \leq \int_{t_1}^{t_2} \int_{SO(3)} \|\Omega\|^2_{M_{3\times 3}(\mathbb{R})} \nu_t(dX)dt \qquad (0 < t_1 < t_2 < 1). \tag{6.5}$$

Then by Theorem 23.9 of Ref.[42], the path (ν_t) of probability measures is absolutely continuous, so there exists $\ell \in L^1[0,1]$ such that $W_2(\nu_{t_2},\nu_{t_1}) \leq \int_{t_1}^{t_2} \ell(t)dt$ and $1/2$-Hölder continuous, so there exists $C > 0$ such that $W_2(\nu_{t_2},\nu_{t_1}) \leq C|t_2 - t_1|^{1/2}$.

Example VI.1. (i) If $\Omega_t \in M_{3\times 3}(\mathbb{R})$ is skew, and X_t, Y_t give solutions of the differential equation

$$\frac{dX}{dt} = \Omega_t X, X(0) = X_0; \quad \frac{dY}{dt} = \Omega_t Y, Y(0) = Y_0 \tag{6.6}$$

then $d(X_t,Y_t) = d(X_0,Y_0)$. We deduce that if X_0 is distributed according to Haar measure on $SO(3)$, then X_t is also distributed according to Haar measure since the measure, the metric and solutions are all preserved via $X \mapsto XU$.

(ii) As an alternative, we can consider X_0 to have first column $[0;0;1]$ and observe the evolution of the first column T of X under the (6.1) where T evolves on $\mathbb{S}^2$.

We now consider the case in which Ω as in (5.2) is a $so(3)$-valued random variable over $(M_\infty, \mu_{K,\beta}, L^2)$.

Proposition VI.2. *Suppose that $\Omega = \Omega(u(\cdot,t))$ where $u(x,t)$ is a solution of NLS and that*

$$\int_{B_K} \|\Omega(u(\cdot,0))\|^2_{M_{3\times 3}(\mathbb{R})} \mu_{K,\beta}(du) \tag{6.7}$$

converges. Then for almost all u with respect to $\mu_{K,\beta}$, there exists a flow $(\nu_t(dX;u))$ of probability measures on $SO(3)$.

Proof. Each solution u of NLS determines Ω so that the associated ODE (6.1) transports the initial distribution of $X_0 \in SO(3)$ to a probability measure on $SO(3)$; then we average over the u with respect to $\mu_K(du)$. This Gibbs measure is invariant under the NLS flow, so by Fubini's theorem

$$\int_{B_K} \int_0^1 \int_{SO(3)} \|\Omega(u(\cdot,t))\|^2_{M_{3\times 3}(\mathbb{R})} \nu_t(dX)dt\mu_K(du) \tag{6.8}$$

converges. Hence the condition (6.3) is satisfied, for almost all u, and we can invoke Theorem 23.9 of Ref.[42]. □

For the finite-dimensional M_n of (4.36) and solutions $u_n = \kappa_n e^{i\sigma_n}$, the modified Hasimoto differential equations are

$$\frac{\partial}{\partial x} X^{(n)}(x,t) = \begin{bmatrix} 0 & \kappa_n & 0 \\ -\kappa_n & 0 & \tau_n \\ 0 & -\tau_n & 0 \end{bmatrix} X^{(n)}(x,t), \tag{6.9}$$

and

$$\frac{\partial}{\partial t}X^{(n)}(x,t)=\begin{bmatrix}0 & -\tau_n\kappa_n & \frac{\partial\kappa_n}{\partial x}\\ \tau_n\kappa_n & 0 & \frac{\partial\sigma_n}{\partial t}+\beta\kappa_n^2\\ -\frac{\partial\kappa_n}{\partial x} & -\frac{\partial\sigma_n}{\partial t}-\beta\kappa_n^2 & 0\end{bmatrix}X^{(n)}(x,t) \tag{6.10}$$

involves $\tau_n=\frac{\partial\sigma_n}{\partial x}$ and $(\frac{\partial\kappa_n}{\partial x})^2+\tau_n^2\kappa_n^2=(\frac{\partial P_n}{\partial x})^2+(\frac{\partial Q_n}{\partial x})^2$ which is continuous, so there exists a solution $X^{(n)}(x,t)\in SO(3)$. We can interpret the solutions as elements of a fibre bundle over $(M_n,\mu_K^{(n)},L^2)$ with fibres that are isomorphic to $SO(3)$.

Let $P+iQ=\kappa e^{i\sigma}$ be a solution of NLS and let

$$\Omega_1=\begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}. \tag{6.11}$$

Proposition VI.3. *(i) Let $P+iQ=\kappa e^{i\sigma}$ be a solution of NLS with initial data in $P(x,0)+iQ(x,0)\in B_K\cap H^1$. Then Ω_1 in (6.11) gives an $so(3)$-valued vector field in $L^2(\kappa^2(x,t)dx)$.*

(ii) Let $P+iQ=\kappa e^{i\sigma}$ be a solution of NLS with initial data $P(x,0)+iQ(x,0)\in H^1\cap B_K$, and let $P_n+iQ_n=\kappa_n e^{i\sigma_n}$ be the corresponding solution of the NLS truncated in Fourier space, giving matrix $\Omega_1^{(n)}$. Let $X_t^{(n)}(x)$ be a solution of (6.9) and suppose that $X^{(n)}$ converges weakly in L^2 to $X_t(x)$. Then X_t gives a weak solution of (5.1).

Proof. (i) With $\omega=\sqrt{\kappa^2+\tau^2}$, we have

$$\exp(h\Omega_1)=I+\frac{\sin h\omega}{\omega}\Omega_1+\frac{1-\cos h\omega}{\omega^2}\Omega_1^2$$

where the entries of Ω_1^2 are bounded by $\kappa^2+\tau^2$, hence

$$\int_{\mathbb{T}}\|\Omega_1(x,t)\|^2_{M_{3\times3}(\mathbb{R})}\kappa(x,t)^2dx<\infty \tag{6.12}$$

for $u\in H^1$; however, there is no reason to suppose that τ itself is integrable with respect to dx.

(ii) By (5.4) and (5.5), we have $\kappa\Omega_1\in L_x^2$ for all $u\in H^1$. Moreover, Bourgain[12] has shown that for initial data $P(x,0)+iQ(x,0)=\kappa(x,0)e^{i\sigma(x,0)}$ in $H^1\cap B_K$, the map

$$\kappa(x,0)e^{i\sigma(x,0)}\mapsto\kappa(x,t)\Omega_1(x,t)\in L^2 \tag{6.13}$$

is Lipschitz continuous for $0\le t\le t_0$ with Lipschitz constant depending upon $t_0,K>0$. We have

$$\begin{aligned}\frac{\|\kappa(x+h,t)X(x+h,t)-\kappa(x,t)X(x,t)\|^2}{h^2}&\le 2\Big(\frac{1}{h}\int_x^{x+h}\big|\frac{\partial\kappa}{\partial y}(y,t)\big|dy\Big)^2\\ &\quad+2\Big(\frac{1}{h}\int_x^{x+h}\kappa(y,t)\|\Omega_1(y,t)\|dy\Big)^2\end{aligned} \tag{6.14}$$

where the right-hand side is integrable with respect to x by the Hardy–Littlewood maximal inequality and (6.12). Suppose that $X^{(n)}$ is a solution of (5.1). We take τ_n to be locally bounded. Then by applying Cauchy–Schwarz inequality to the integral

$$X^{(n)}(x+h,t)-X^{(n)}(x,t)=\int_0^h\Omega_1^{(n)}(x+s,t)X^{(n)}(x+s,t)ds,$$

we deduce that

$$\begin{aligned}&\int_{[0,2\pi]}\|X^{(n)}(x+s,t)-X^{(n)}(x,t)\|^2_{M_{3\times3}(\mathbb{R})}\kappa_n(x,t)^2dx\\ &\quad\le h\int_0^h\int_{[0,2\pi]}\|\Omega_1^{(n)}(x+s,t)\|^2_{M_{3\times3}(\mathbb{R})}\kappa_n(x,t)^2dxds\end{aligned} \tag{6.15}$$

where the integral is finite by (6.12). Also

$$\sum_{j=1}^N\frac{\|X^{(n)}(x_j,t)-X^{(n)}(x_{j-1},t)\|^2_{M_{3\times3}(\mathbb{R})}}{x_j-x_{j-1}}\le\int_{x_0}^{x_N}\|\Omega_1^{(n)}(x,t)\|^2dx$$

for $0 < x_1 < x_2 < \cdots < x_N < 2\pi$. We have

$$\frac{\partial}{\partial x}\big(\kappa_n X^{(n)}\big) = \frac{\partial \kappa_n}{\partial x} X^{(n)} + \kappa^{(n)}\Omega_1^{(n)} X^{(n)} \tag{6.16}$$

so for $Z \in C^\infty([0,2\pi]; M_{3\times 3}(\mathbb{R}))$ and the inner product on $M_{3\times 3}(\mathbb{R})$, we have

$$\begin{aligned}&\langle \kappa_n(2\pi)X^{(n)}(2\pi), Z(2\pi)\rangle - \langle \kappa_n(0)X^{(n)}(0), Z(0)\rangle - \int_0^{2\pi} \kappa_n(x)\langle X^{(n)}(x), Z(x)\rangle\, dx\\ &= \int_0^{2\pi} \frac{\partial \kappa_n}{\partial x}\langle X^{(n)}(x), Z(x)\rangle\, dx + \int_0^{2\pi} \langle X^{(n)}, \kappa_n(x)\Omega_1^{(n)}(x)^\top Z(x)\rangle dx\end{aligned} \tag{6.17}$$

where $\kappa_n \to \kappa$ in H^1, so with norm convergence, we have $\frac{\partial \kappa_n}{\partial x} \to \frac{\partial \kappa}{\partial x}$ in L^2, and $\kappa_n \Omega^{(n)} \to \kappa\Omega_1$ as $n \to \infty$, and with weak convergence in L^2, we have $X^{(n)} \to X$, so

$$\begin{aligned}&\langle \kappa(2\pi)X(2\pi), Z(2\pi)\rangle - \langle \kappa(0)X(0), Z(0)\rangle - \int_0^{2\pi} \kappa(x)\langle X(x), Z(x)\rangle\, dx\\ &= \int_0^{2\pi} \frac{\partial \kappa}{\partial x}\langle X(x), Z(x)\rangle\, dx + \int_0^{2\pi} \langle X, \kappa(x)\Omega_1(x)^\top Z(x)\rangle dx.\end{aligned} \tag{6.18}$$

□

The simulation of this differential equation computes $X_x \in \mathbb{S}^2$ starting with $X_0 = [0;0;1]$ and produces a frame $\{X_x, \Omega_x X_x, X_x \times \Omega_x X_x\}$ of orthogonal vectors. Geodesics on $\mathbb{S}^2$ are the curves such that the principal normal is parallel to the position vector, namely the great circles. For a geodesic, $X_x \times \Omega_x X_x$ is perpendicular to the plane that contains the great circle.

Let $P + iQ = \kappa e^{i\sigma}$ be a solution of NLS and let

$$\Omega_2 = \begin{bmatrix} 0 & -\kappa\tau & \frac{\partial\kappa}{\partial x}\\ \kappa\tau & 0 & 0\\ -\frac{\partial\kappa}{\partial x} & 0 & 0\end{bmatrix}. \tag{6.19}$$

Proposition VI.4. *(i) Let $P + iQ = \kappa e^{i\sigma}$ be a solution of NLS with initial data $P(x,0) + iQ(x,0) \in B_K$. Then $x \mapsto \int_0^x \Omega_2(y,t)dy$ gives a $so(3)$-valued stochastic of finite quadratic variation on $[0,2\pi]$ almost surely with respect to $\mu_K(dPdQ)$.*

(ii) Let $P + iQ = \kappa e^{i\sigma}$ be a solution of NLS with initial data $P(x,0) + iQ(x,0) \in H^1 \cap B_K$, and let $P_n + iQ_n = \kappa_n e^{i\sigma_n}$ be the corresponding solution of the NLS truncated in Fourier space, giving matrix $\Omega_2^{(n)}$. Let $X_t^{(n)}$ be a solution of (6.10). Then $X_t^{(n)}$ converges in L_x^2 norm to X_t as $n \to \infty$ where X_t gives a weak solution of (5.2).

Proof. (i) The essential estimate is

$$\begin{aligned}&\int_{B_K} \sum_j |\kappa(x_{j+1},t) - \kappa(x_j,t)|^2 \mu_K(du)\\ &\le \sum_j \Big(\int_{B_K} |u(x_{j+1},t) - u(x_j,t)|^2 \mu_K(du)\Big)\\ &\le \sum_j \Big(\int_{B_K} |u(x_{j+1},t) - u(x_j,t)|^4 W_K(du)\Big)^{1/2}\Big(\int_{B_K}\Big(\frac{d\mu_K}{dW}\Big)^2 dW\Big)^{1/2}\\ &\le C\sum_j \Big(\int_{B_K} |u(x_{j+1},t) - u(x_j,t)|^2 W(du)\Big)^{1/2}\\ &\le C\sum_j (x_{j+1} - x_j) \le 2\pi C.\end{aligned} \tag{6.20}$$

The function σ is a progressively measurable stochastic process adapted with respect to a suitable filtration, and with differential satisfying an Ito integral equation[22]. Therefore, we can control the $\kappa\tau$ term via

$$\int_0^x (\kappa d\sigma - 2^{-1}\kappa^2\langle d\sigma, d\sigma\rangle) = \int_0^x \kappa\nabla\sigma\cdot\begin{bmatrix} dP\\ dQ\end{bmatrix} = \int_0^x \frac{-QdP + PdQ}{\sqrt{P^2+Q^2}} \tag{6.21}$$

which is a bounded martingale transform of Wiener loop. This formula is reminiscent of Levy's stochastic area as in Example 5.1 of Ref.[26].

(ii) By (5.4) and (5.5), we have $\Omega_2 \in L^2_x$ for all $u \in H^1$. Bourgain[12] has shown that for initial data $P(x,0)+iQ(x,0) = \kappa(x,0)e^{i\sigma(x,0)}$ in $H^1 \cap B_K$, the map

$$\kappa(x,0)e^{i\sigma(x,0)} \mapsto \Omega_2(x,t) \in L^2_x \tag{6.22}$$

is Lipschitz continuous for $0 \le t \le t_0$ with Lipschitz constant depending upon $t_0, K > 0$. We have

$$\int_0^{2\pi} \|\Omega_2(x)\|^2 dx \le 2\int_0^{2\pi} \Big(\Big(\frac{\partial \kappa}{\partial x}\Big)^2 + \kappa(x)^2\tau(x)^2 + \kappa(x)^4\Big)dx,$$

where the final integral is part of the Hamiltonian. With $Z \in C^\infty(\mathbb{T}; M_{3\times 3}(\mathbb{R}))$, we have the integral equation for the pairing $\langle\cdot,\cdot\rangle$ on $L^2([0,2\pi], M_{3\times 3}(\mathbb{R}))$

$$\langle X_t^{(n)}, Z\rangle = \langle X_0^{(n)}, Z\rangle + \int_0^t \langle X_s^{(n)}, (\Omega_s^{(n)})^\top Z\rangle\, ds. \tag{6.23}$$

Consider the variational differential equation in $L^2([0,2\pi], M_{3\times 3}(\mathbb{R}))$

$$\begin{aligned}\frac{d}{dt}(X^{(m)}(x,t) - X^{(n)}(x,t)) &= \Omega_2^{(n)}(x,t)(X^{(m)}(x,t) - X^{(n)}(x,t)) \\ &\quad + (\Omega_2^{(m)}(x,t) - \Omega_2^{(n)}(x,t))X^{(m)}(x,t)\end{aligned} \tag{6.24}$$

where $\Omega_2^{(n)}(x,t)$ and $\Omega_2^{(m)}(x,t) - \Omega_2^{(n)}(x,t)$ are skew.

We introduce a family of matrices $U^{(n)}(x;t,s)$ such that $U^{(n)}(x;t,r)U^{(n)}(x;r,s) = U^{(n)}(x;t,s)$ for $t > r > s$ and $U^{(n)}(x;t,t) = I$ such that

$$\frac{\partial}{\partial t}U^{(n)}(x;t,s) = \Omega_2^{(n)}(x;t)U^{(n)}(x;t,s). \tag{6.25}$$

Then the variational equation has solution

$$\begin{aligned}X^{(m)}(x,t) - X^{(n)}(x,t) &= U^{(n)}(x;t,0)(X^{(m)}(x,0) - X^{(n)}(x,0)) \\ &\quad + \int_0^t U^{(n)}(x;t,r)(\Omega_2^{(m)}(x;r) - \Omega_2^{(n)}(x;r))X^{(m)}(x,r)dr.\end{aligned}$$

Then

$$\begin{aligned}&\frac{d}{dt}\langle X^{(m)}(t) - X^{(n)}(t), X^{(m)}(t) - X^{(n)}(t)\rangle_{L^2_x} \\ &= 2\,\mathrm{Re}\big\langle(\Omega_2^{(m)}(t) - \Omega_2^{(n)}(t))X^{(m)}(t), X^{(m)}(t) - X^{(n)}(t)\big\rangle_{L^2_x} \\ &\le \|\Omega_2^{(m)}(t) - \Omega_2^{(n)}(t)\|^2_{L^2_x}\|X^{(m)}(t)\|^2_{L^2_x} + \|X^{(m)}(t) - X^{(n)}(t)\|^2_{L^2_x}\end{aligned} \tag{6.26}$$

so from this differential inequality we have

$$\|X^{(m)}(t) - X^{(n)}(t)\|^2_{L^2_x} \le e^t\|X^{(m)}(0) - X^{(n)}(0)\|^2_{L^2_x} + \int_0^t e^{t-s}\|\Omega_2^{(m)}(s) - \Omega_2^{(n)}(s)\|^2_{L^2_x}ds. \tag{6.27}$$

Now $X^{(m)}(0) - X^{(n)}(0) \to 0$ and $\Omega_2^{(m)}(s) - \Omega_2^{(n)}(s) \to 0$ in L^2_x norm as $n,m \to \infty$, so there exists $X(x,t) \in L^2_x$ such that $X(x,t) - X^{(n)}(x,t) \to 0$ in L^2_x norm as $n \to \infty$.

We deduce that

$$\langle X(t), Z\rangle_{L^2_x} = \langle X_0, Z\rangle_{L^2_x} + \int_0^t \langle X_u, (\Omega_2(u))^\top Z\rangle_{L^2_x}\, du, \tag{6.28}$$

so we have a weak solution of the ODE. □

Let $\Omega_2^{(n,u_n)}(x,t)$ be the Fourier truncated matrix that corresponds to a solution u_n of the Fourier truncated equation NLS_n, then let $X^{n,u_n}(x,t)$ be the solution of the ODE (6.10). By Proposition VI.2, the map $u_n \mapsto X^{n,u_n}(\cdot,t)$ pushes forward the modified Gibbs measure $\mu_K^{(n)}$ to a measure on $(C(M_n;SO(3)), L^2)$ that satisfies a Gaussian concentration of measure inequality with constant $\alpha(\beta,K)/n^2$; compare (3.9).

Corollary VI.5. *For each $Z \in L^2([0,2\pi]; M_{3\times 3}(\mathbb{R}))$, introduce the $\mathbb{R}$-valued random variable on $(M_n, L^2, \mu_K^{(n)})$ by*

$$Z_n(u_n) = \int_{[0,2\pi]} \langle X^{(n,u_n)}(x,t), Z(x)\rangle dx. \tag{6.29}$$

(i) Then the distribution $\nu^{(n)}$ of Z_n satisfies the Gaussian concentration inequality

$$\int_{M_n} \exp\Big(tZ_n - t\int_{M_n} Z_n d\mu_K^{(n)}\Big)\mu_K^{(n)}(du_n) \le \exp\big(n^2t^2/\alpha(\beta,K)\big) \qquad (t\in\mathbb{R}). \tag{6.30}$$

(ii) Let $\nu_N^{(n)} = N^{-1}\sum_{j=1}^N \delta_{Z_n^{(j)}}$ be the empirical distribution of N independent copies of Z_n. Then $W_1(\nu_N^{(n)}, \nu^{(n)}) \to 0$ almost surely as $N\to\infty$.

Proof. (i) As with u_n, we introduce the corresponding data for another solution v_n. As in (6.27), we have

$$\begin{aligned}\|X^{(n,u_n)}(x,t) - X^{(n,v_n)}(x,t)\|^2_{\mathbb{R}^3} &\le e^t\|X^{(n,u_n)}(0) - X^{(n,v_n)}(0)\|^2_{\mathbb{R}^3}\\ &\quad + \int_0^t e^{t-s}\|\Omega_2^{(n,u_n)}(x,s) - \Omega_2^{(n,v_n)}(x,s)\|^2_{so(3)}ds.\end{aligned} \tag{6.31}$$

For given initial condition $X^{n,v_n)}(0) = X^{(n,u_n)}(0)$, and $T>0$, we can take the supremum over t, then integrate this with respect to x and obtain

$$\begin{aligned}&\int_0^{2\pi} \sup_{0<t<T} \|X^{(n,u_n)}(x,t) - X^{(n,v_n)}(x,t)\|^2_{\mathbb{R}^3}dx\\ &\qquad \le e^T\int_0^T \|\Omega_2^{(n,u_n)}(x,s) - \Omega_2^{(n,v_n)}(x,s)\|^2_{L^2_x}ds\end{aligned} \tag{6.32}$$

so $\Omega^{(u)} \mapsto X^u$ is a Lipschitz function $L^2([0,2\pi]\times[0,T], so(3)) \to L^2([0,2\pi]; L^\infty([0,T],\mathbb{R}^3))$. By Bourgain's results, there exists $C>0$ such that

$$\begin{aligned}\|\Omega_2^{(n,u_n)}(x,s) - \Omega_2^{(n,v_n)}(x,s)\|_{L^2_x} &\le C\|u_n(x,s) - v_n(x,s)\|_{H^1_x}\\ &\le Cn\|u_n(x,0) - v_n(x,0)\|_{L^2_x},\end{aligned} \tag{6.33}$$

so $u_n \mapsto X^{(n,u_n)}$ is a Lipschitz function on L^2_x, albeit with a constant growing with n. Thus we can push forward the modified Gibbs measure $(M_n, L^2, \mu_{K,\beta}^{(n)}) \to L^2([0,2\pi]; M_{3\times 3}(\mathbb{R}))$ so that the image measure satisfies a Gaussian concentration inequality with constant $\alpha(\beta,K)/n^2$ dependent upon n. For each $Z\in L^2([0,2\pi]; M_{3\times3}(\mathbb{R}))$, we introduce Z_n, so that where $u_n\mapsto Z_n$ is Cn-Lipschitz function from $(M_n, L^2, \mu_{K,\beta}^{(n)})$ to $\mathbb{R}$. The random variable Z_n therefore satisfies the Gaussian concentration inequality (6.30).

(ii) By Theorem IV.3, we can use the Borel–Cantelli Lemma to show that

$$\mathbb{P}\Big[\big|W_1(\nu_N^{(n)},\nu^{(n)}) - \mathbb{E}W_1(\nu_N^{(n)},\nu^{(n)})\big| > \varepsilon \quad \text{for infinitely many} \quad N\Big] = 0 \qquad (\varepsilon>0),$$

where by Proposition IV.4, $\mathbb{E}W_1(\nu_N^{(n)},\nu^{(n)}) \to 0$ as $N\to\infty$. □

Consider a coupling of $(M_n, L^2, \mu_{K,\beta}^{(n)})$ and $(M_\infty, L^2, \mu_{K,\beta})$ involving measure π_n. For any bounded continuous $\varphi:\mathbb{C}\to\mathbb{R}$ we can consider

$$\begin{aligned}&\int_{M_n}\varphi(Z_n(u_n))\mu_{K,\beta}^{(n)}(du_n) - \int_{M_\infty}\varphi(Z(u))\mu_{K,\beta}(du)\\ &\qquad = \iint_{M_n\times M_\infty}\big(\varphi(Z_n(u_n)) - \varphi(Z(u))\big)\pi_n(du_n du)\end{aligned} \tag{6.34}$$

where

$$\mathscr{D}_{L^2}(\hat{M}_n,\hat{M}_\infty)^2 = \iint_{M_n\times M_\infty}\delta(u_n,u)^2\pi_n(du_n du) \to 0 \qquad (n\to\infty).$$

Proposition VI.6. *Let $(\varphi_j)_{j=1}^\infty$ be a dense sequence in $Ball(C_c(\mathbb{C};\mathbb{R}))$ and $(Y_\ell)_{\ell=1}^\infty$ a dense sequence in $Ball(L^2)$. Then there exists a subsequence (n_k) such that*

$$\int_{M_{n_k}} \varphi_j\big(\langle X^{(n_k,u_{n_k})}, Y_\ell\rangle\big)\mu_{K,\beta}^{(n_k)}(du_{n_k}) \tag{6.35}$$

converges as $n_k \to \infty$ for all $j, \ell \in \mathbb{N}$.

Proof. We can introduce a metric so that $M = \prod_{n=1}^\infty M_n \sqcup M_\infty$ becomes a complete and separable metric space, and we can transport μ_n onto M. Then $\omega = 2^{-1}\mu_\infty + \sum_{n=1}^\infty 2^{-n-1}\mu_n$ is a probability measure on M, and μ_n is absolutely continuous with respect to ω, so $d\mu_n = f_n d\omega$ for some probability density function $f_n \in L^1(\omega)$. By convergence in total variation from Lemma IV.7 (ii), here exists $f_\infty \in L^1(\omega)$ such that $f_n \to f_\infty$ in L^1 as $n \to \infty$. Given a bounded sequence $(g_n)_{n=1}^\infty$ in $L^\infty(\omega)$, there exists $g_\infty \in L^\infty(\omega)$ and a subsequence (n_k) such that

$$\int g_{n_k} d\mu_{n_k} = \int g_{n_k} f_{n_k} d\omega \to \int g_\infty f_\infty d\omega = \int g_\infty d\mu_\infty. \tag{6.36}$$

□

Remark VI.7. For $u \in M_\infty$, we have $u_n = D_n u \in M_n$ so that $u_n \to u$ in L^2 norm as $n \to \infty$. It is plausible that (6.34) tends to 0 as $n \to \infty$, but we do not have a proof. Unfortunately, the constants are not sharp enough to allow us to use Proposition IV.8 to deduce W_2 convergence for the distributions on $SO(3)$.

VII. EXPERIMENTAL RESULTS

Our objective in this section is to obtain a (random) numerical approximation to the solution of (6.9). We consider the case where the parameter β in (1.3) is equal to 0. Note that in this case, the Gibbs measure reduces to Wiener loop measure and stochastic processes with the Wiener loop measure as their law are by definition Brownian loop. Equation (6.9) is a PDE with respect to the space variable x, while the parameter of a stochastic process in an SDE is colloquially referred to as time. To avoid confusion, in this section we refer to x as s; whereas the time variable t is suppressed.

Recall the polar decomposition $P + iQ = \kappa e^{i\sigma}$ where, $\kappa = \sqrt{P^2+Q^2}$ and σ is such that $\tau = \frac{\partial\sigma}{\partial s}$. Define $\sigma_\varepsilon(P,Q) := \tan^{-1}(\frac{PQ}{P^2+\varepsilon^2})$ as the regularised Itô integral of τ. The Itô differential can be written as

$$d\sigma_\varepsilon = f_1(P,Q)dP + f_2(P,Q)dQ + f_3(P,Q)ds,$$

where

$$\begin{aligned}
f_1(P,Q) &:= \frac{(\varepsilon^2 - P^2)Q}{(\varepsilon^2+P^2)^2 + P^2Q^2},\\
f_2(P,Q) &:= \frac{P(\varepsilon^2+P^2)}{(\varepsilon^2+P^2)^2+P^2Q^2},\\
f_3(P,Q) &:= -\frac{2P^3Q(\varepsilon^2+P^2)}{((\varepsilon^2+P^2)^2+P^2Q^2)^2} - \frac{2PQ\left((\varepsilon^2+P^2)^2+P^2Q^2\right)}{((\varepsilon^2+P^2)^2+P^2Q^2)^2} - \frac{(\varepsilon^2-P^2)Q\left(2PQ^2+4P(\varepsilon^2+P^2)\right)}{((\varepsilon^2+P^2)^2+P^2Q^2)^2}.
\end{aligned}$$

We can write (6.9) in the form of a SDE, including a correction to convert from a Stranovich SDE into an Itô SDE as follows

$$dX_s = \mathbf{A}X_s ds + \mathbf{B}X_s dP + \mathbf{C}X_s dQ \tag{7.1}$$

where,

$$\begin{aligned}
\mathbf{A} &= \begin{bmatrix} 0 & \sqrt{P^2+Q^2} & 0 \\ -\sqrt{P^2+Q^2} & \frac{1}{2}f_1^2(P,Q) + \frac{1}{2}f_2^2(P,Q) & f_3(P,Q) \\ 0 & -f_3(P,Q) & \frac{1}{2}f_1^2(P,Q)+\frac{1}{2}f_2^2(P,Q) \end{bmatrix},\\
\mathbf{B} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & f_1(P,Q) \\ 0 & -f_1(P,Q) & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & f_2(P,Q) \\ 0 & -f_2(P,Q) & 0 \end{bmatrix}.
\end{aligned} \tag{7.2}$$

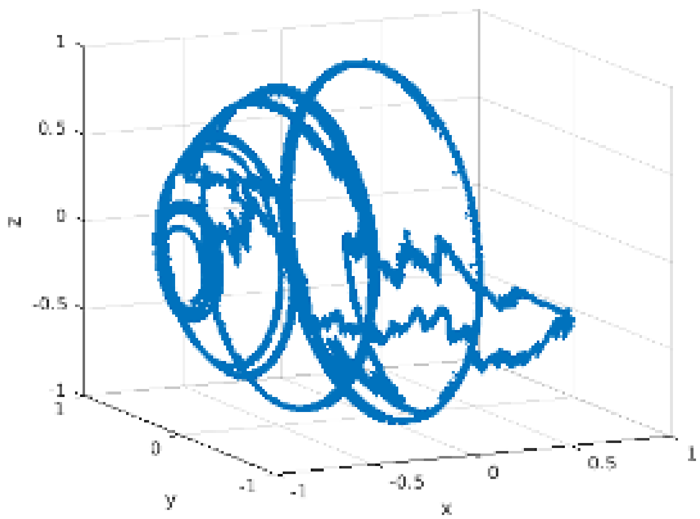

FIG. 1. The figure demonstrates a sample path of the stochastic process X_s, which is a solution to Equation (7.3). As $X_s \in SO(3)$ the path is visualised as the action of X_s applied to a unit vector in $\mathbb{R}^3$. The numerical solution shown is for $s \in [0,10]$, and has a step size of $h = 10^{-5}$.

As justified above P and Q are each a Brownian bridge with period $T = 2\pi$, thus they can be expressed in terms of Brownian motions W_1 and W_2; that is, $P(s) = W_1(s) - sW_1(2\pi)/2\pi$ and likewise for Q. Equation (7.1) is now written as a standard Itô SDE,

$$dX_s = \left(\mathbf{A} + \frac{W_1(2\pi)}{2\pi}\mathbf{B} + \frac{W_2(2\pi)}{2\pi}\mathbf{C}\right)X_s ds + \mathbf{B}X_s dW_1 + \mathbf{C}X_s dW_2 \tag{7.3}$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are defined as in Equation (7.2). The resulting stochastic process $X_s \in SO(3)$ is then used to rotate the unit vector $y_0 = [0,0,1]^\top$ on $\mathbb{S}^2$ to $y_s = X_s y_0$, the third column of X_s. The sample paths of this process can be described by construction of a frame $\{y_s, y_s', y_s \times y_s'\}$. In order to simulate this SDE, we make use of a numerical scheme for matrix SDEs in $SO(3)$ developed by Marjanovic and Solo[34]. This involves a single step geometric Euler-Maruyama method, called g-EM, in the associated Lie algebra. Figure 1 demonstrates a sample-path of y_s generated via this method, and the code used to simulate a sample path is available[30]. The sample paths start off on the great circle perpendicular to the y-axis, and so have constant binormal $y_s \times y_s'$. As a sample path extends past the great circle, the binormal vector at each point deviates slowly; thus a sample path can be thought of as a precessing orbit.

The Itô process y_s is derived from the solution to Equation (7.3) and takes values in $\mathbb{R}^3$. Let $\hat{y}_{s,h}$ denote the numerical approximation to y_s on $[0,T]$ with step size h, which is calculated using the g-EM method. The approximation error converges to zero in the L^2 space of Itô processes as the step size $h \to 0$,

$$\mathbb{E}\left[\sup_{0\le s<T} \|y_s - \hat{y}_{s,h}\|^2_{\mathbb{R}^3}\right] = \mathscr{O}(h^{1-\varepsilon}), \tag{7.4}$$

for some $\varepsilon > 0$ (See Piggott[39]). A value of $\varepsilon = 1/4$ allows us to maintain control of the implied constants on the interval $[0,1]$, and h is taken to be 10^{-5}. We apply g-EM to Equation (7.3) on the interval $[0,10]$, upon which a smaller value of h would be welcomed. However, we are attempting to calculate a distribution, so we need a large number of sample-paths.

The computational complexity of simulating a single sample-path is $\mathscr{O}(T/h)$ where T denotes the length of the interval simulated. Therefore, for a total of N samples, the computational complexity of our simulation algorithm is $\mathscr{O}(NT/h)$. We run our simulations using a machine equipped with an 8-core Intel Xeon Gold 6248R CPU with a clock speed of 2993 Mhz; we take advantage of integrated parallelisation in MATLAB. With $h = 10^{-5}$ and $N = 2\times 10^6$ the algorithm takes around 1 week to run on our system.

Since the sample paths are constrained to $\mathbb{S}^2$ the points y_s can be specified in spherical coordinates of longitude $\theta_s \in [-\pi,\pi)$ and colatitude $\phi_s \in [0,\pi]$. Figure 2 demonstrates the empirical joint distribution of θ_s and ϕ_s for two different values of s. As can be observed, the distribution of (θ_s,ϕ_s) varies with s. We hypothesise that the angles θ_s and ϕ_s evolve to become statistically independent, and that y_s will eventually be uniformly distributed on the sphere. In the remainder of the section, we test this hypothesis statistically.

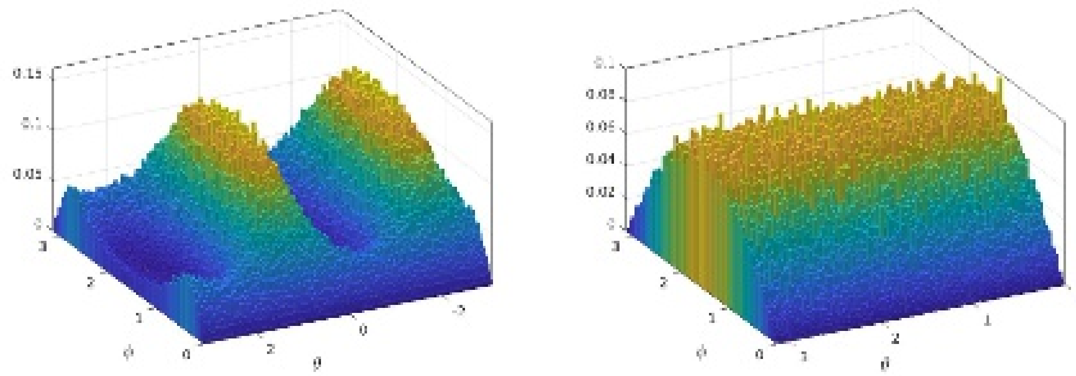

FIG. 2. Histograms of the joint distribution of the third column of X_s at two timesteps, $s=1$ (**left**) and $s=10$ (**right**). The distribution lies on the sphere $\mathbb{S}^2$ and thus the axes are chosen as the longitude θ_s and colatitude ϕ_s. With respect to these axes the marginals of the distribution become more independent over time.

a. ***Wasserstein distance between measures on*** $\mathbb{S}^2$**.** We start by calculating the Wasserstein distance $W_1(\nu_1,\nu_2)$ between probability measures ν_1 and ν_2 on $\mathbb{S}^2$, which are absolutely continuous with respect to area and have disintegrations

$$d\nu_j = f_j(\theta)g_j(\phi \mid \theta)\sin\phi\, d\phi d\theta \qquad (\theta\in[-\pi,\pi],\phi\in[0,\pi], j=1,2)$$

where f_j $(j=1,2)$ are probability density functions on $[-\pi,\pi]$ that give the marginal distributions of ν_j in the longitude θ variable, and g_j in the colatitude variable. Let F_j be the cumulative distribution function of $f_j(\theta)d\theta$ and G_j be the cumulative distribution function of $g_j(\phi)\sin\phi\, d\phi$. We measure $W_1(\nu_1,\nu_2)$ in terms of one-dimensional distributions. Given distributions on $\mathbb{R}$ with cumulative distribution functions F_1 and F_2, we write $W_1(F_1,F_2)$ for the Wasserstein distance between the distributions for cost function $|x-y|$. Let $\psi:[-\pi,\pi]\to[-\pi,\pi]$ be an increasing function that induces $f_2(\theta)d\theta$ from $f_1(\theta)d\theta$; then

$$W_1(\nu_1,\nu_2)\le W_1\big(F_1,F_2\big)+\int_{-\pi}^{\pi}W_1\big(G_2(\cdot\mid\psi(\theta)),G_1(\cdot\mid\theta)\big)f_1(\theta)d\theta.$$

In particular, for $f_1(\theta)=1/(2\pi)$ and $g_1(\phi)=1/2$, we have a product measure $\nu_1(d\theta d\phi)=(4\pi)^{-1}\sin\phi d\phi d\theta$ giving normalized surface area on the sphere. Then $F_1(\theta)=(\theta+\pi)/(2\pi)$ and $F_2(\psi(\theta))=(\theta+\pi)/(2\pi)$, so $\psi(2\pi(\tau-1/2))$ for $\tau\in[0,1]$ gives the inverse function of F_2. We deduce that

$$W_1(F_1,F_2)=\int_{-\pi}^{\pi}\Big|\frac{\theta+\pi}{2\pi}-F_2(\theta)\Big|d\theta \tag{7.5}$$

and

$$W_1\big(G_2(\cdot\mid\psi(\theta)),G_1(\cdot\mid\theta)\big)=\int_0^{\pi}\Big|\int_0^{\phi}\big(g_2(\phi'\mid\psi(\theta))-(1/2)\big)\sin\phi' d\phi'\Big|d\phi \tag{7.6}$$

Hence the Wasserstein distance can be bounded in terms of the cumulative distribution functions by

$$\begin{aligned}W_1(\nu_1,\nu_2)&\le W_1(F_2,F_1)+W_1(G_2,G_1)+\int_{-\pi}^{\pi}W_1(G_2(\cdot\mid\theta),G_2)dF_1(\theta)\\&=\int_{-\pi}^{\pi}\Big|\frac{\theta+\pi}{2\pi}-F_2(\theta)\Big|d\theta+\int_0^{\pi}\Big|G_2(\phi)-\frac{1-\cos\phi}{2}\Big|d\phi+\int_{-\pi}^{\pi}\int_0^{\pi}\big|G_2(\phi\mid\theta)-G_2(\phi)\big|dF_1(\theta)d\phi\end{aligned} \tag{7.7}$$

where we have used the triangle inequality to obtain a more symmetrical expression involving the Wasserstein distances for the marginal distributions and the G conditional distributions, namely the dependence of the colatitude distribution on longitude.

For each $s\in[0,10]$, let F^{θ_s} and G^{ϕ_s} be the marginal CDFs of θ_s and ϕ_s respectively. For $N\in\mathbb{N}$, denote by $F_N^{\theta_s}$ and $G_N^{\phi_s}$ the empirical CDFs of θ_s and ϕ_s. We generate empirical CDFs $F_N^{\theta_s}$ and $G_N^{\phi_s}$ with $s=0.3,0.6,0.9,\dots,6.0$, and $N=10^5$. Figure 3 demonstrates that $W_1(F_1,F_N^{\theta_s})$ and $W_1(G_1,G_N^{\phi_s})$, each decreases with increasing s. As a consequence of Theorem IV.3 and Proposition IV.4 for $N=10^5$ with probability at least 0.99 it holds that $W_1(F_N^{\theta_s},F^{\theta_s})\le 0.025$ and $W_1(G_N^{\phi_s},G^{\phi_s})\le 0.018$. Thus, we observe that F^{θ_s} converges to F_1 and G^{ϕ_s} converges to G_1.

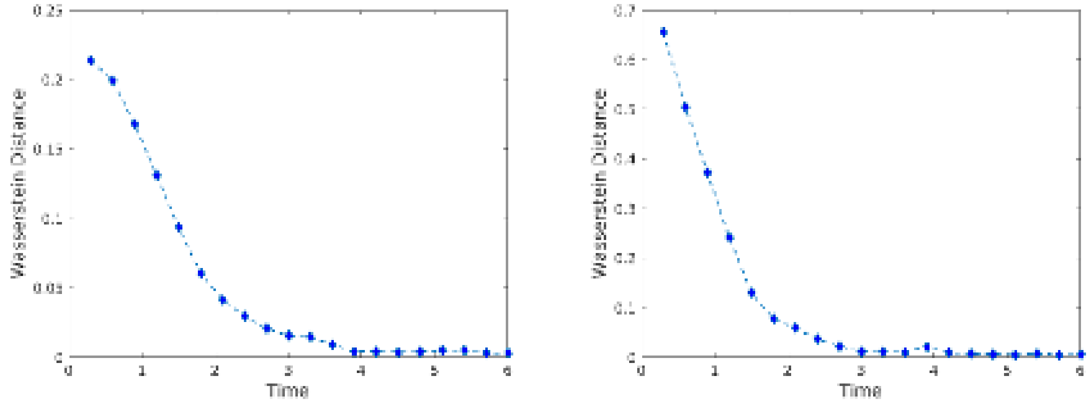

FIG. 3. The plots involve the difference between the CDFs of two marginals. For θ_s, the predicted CDF $F_1(\theta) = (\theta+\pi)/(2\pi)$ is compared with the empirical CDF, $F_N^{\theta_s}$. The Wasserstein distance between $F_1(\theta)$ and $F_N^{\theta_s}$ is displayed on the **left**. For ϕ_s, the predicted CDF $G_1(\phi) = (1-\cos(\phi))/2$ is compared with the empirical CDF $G_N^{\phi_s}$. The Wasserstein distance between $G_1(\phi)$ and $G_N^{\phi_s}$ is displayed on the **right**. The emprical measures considered are created using $N = 10^5$ samples and evaluated at each of the datapoints indicated on the graphs ($s = 0.3, 0.6, ..., 6.0$).

b. ***Hypothesis tests for independence and goodness-of-fit.*** We run a total of 22 hypothesis tests to examine the evolution of the joint distribution of the angles θ_s and ϕ_s. In order to account for multiple testing, we set the significance level of each test to 0.00045, leading to an overall level of 0.01. First, we generate sample paths to obtain $N = 10^5$ realisations of (θ_s, ϕ_s) for each value of $s = 0.3, 0.6, 0.9, \dots, 6.0$. For each s, we test the null hypothesis $H_{0,s}$ that the angles θ_s and ϕ_s are statistically independent, against the alternative hypothesis $H_{1,s}$ that they are dependent. To this end, we rely on a widely used nonparametric independence test, which is based on the Hilbert-Schmidt Independence Criterion (HSIC) dependence measure[2,24]; the implementation is due to Jitkrittum[27]. It is observed that while the null hypothesis is rejected for $s = 0.3, \dots, 2.1$, the test is unable to reject $H_{0,s}$ from $s = 2.4, \dots, 6.0$ at (an overall) significance level 0.01.

We run two Kolmogorov-Smirnov goodness-of-fit tests for $s = 10$ as follows. The first tests the null hypothesis $H_0^{\theta_s}$ that θ_s is distributed according to F_1 against the alternative that it is not; the second tests the null hypothesis $H_0^{\phi_s}$ that ϕ_s is distributed according to G_1 against the alternative that it is not. At significance level 0.01, the tests are unable to reject the null hypotheses $H_0^{\theta_s}$ and $H_0^{\phi_s}$.

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